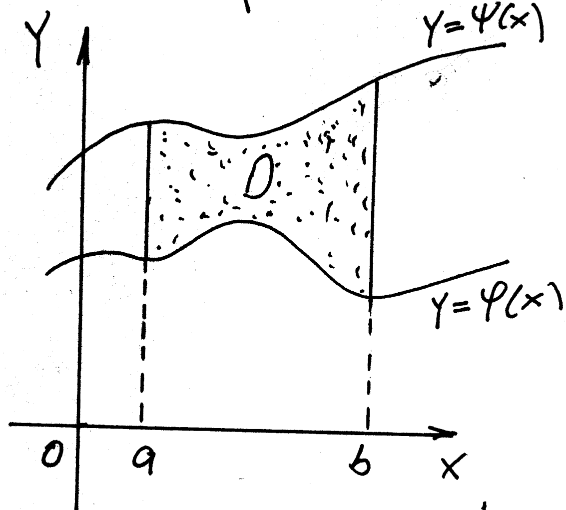
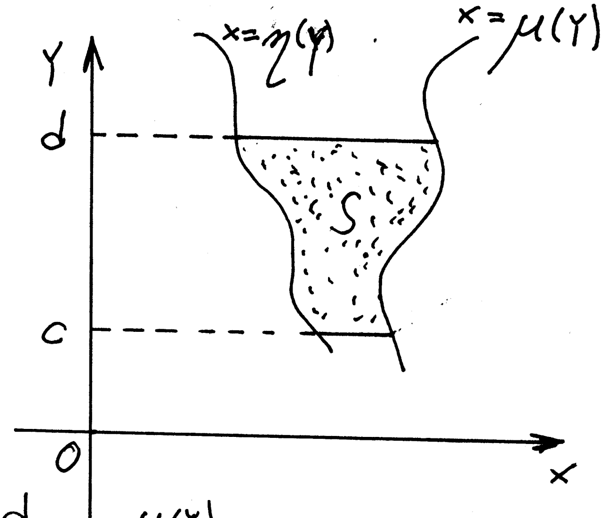
Dvostruki integral

$$\iint_D f(x,y)\,dx\,dy = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy = \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right] dx$$



D - oblast integracije



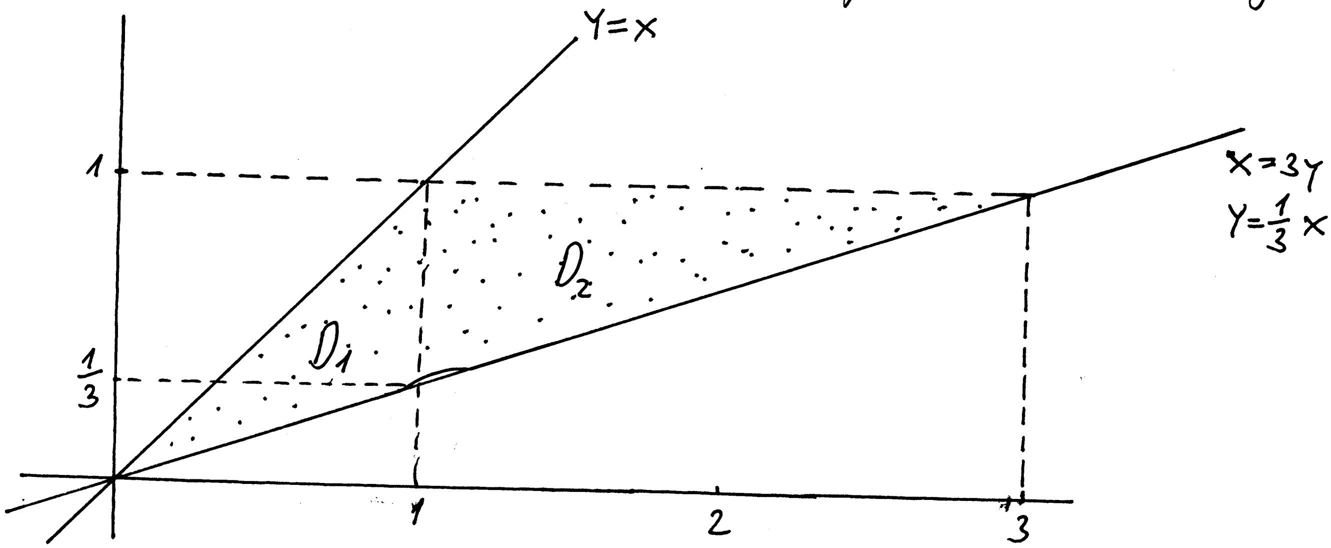
$$\iint_S f(x,y)\,dx\,dy = \int_c^d dy \int_{\eta(y)}^{\mu(y)} f(x,y)\,dx = \int_c^d \left[\int_{\eta(y)}^{\mu(y)} f(x,y)\,dx\right] dy$$

Izmjeniti poredak integracije u integralu

$$I = \int_0^1 dy \int_y^{3y} f(x,y)\,dx \, .$$

Rj.

$x = 3y$ i $x = y$ su prave. Skicirajmo oblast integracije



$$D: \begin{cases} 0 \le y \le 1 \\ y \le x \le 3y \end{cases} = D_1 \cup D_2 \qquad D_1: \begin{cases} 0 \le x \le 1 \\ \frac{1}{3}x \le y \le x \end{cases}$$

$$D_2: \begin{cases} 1 \le x \le 3 \\ \frac{1}{3}x \le y \le 1 \end{cases}$$

$$\int_0^1 dy \int_y^{3y} f(x,y)\,dx = \int_0^1 dx \int_{\frac{1}{3}x}^{x} f(x,y)\,dy + \int_1^3 dx \int_{\frac{1}{3}x}^{1} f(x,y)\,dy$$

Izmeniti poredak integracije u integralu

$$I=\int_0^1 dx\int_x^{\sqrt{2-x^2}} f(x,y)\,dy .$$

Rj.

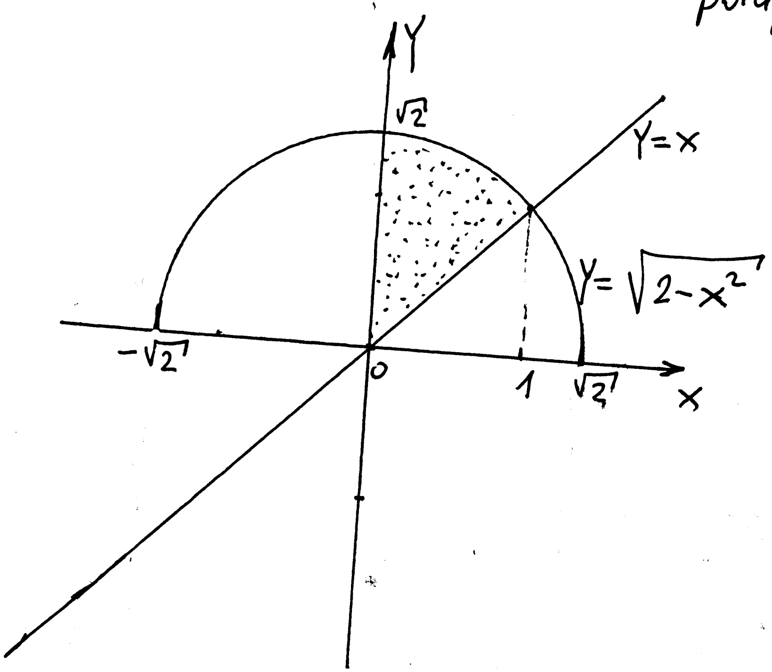
$y=x$ prava

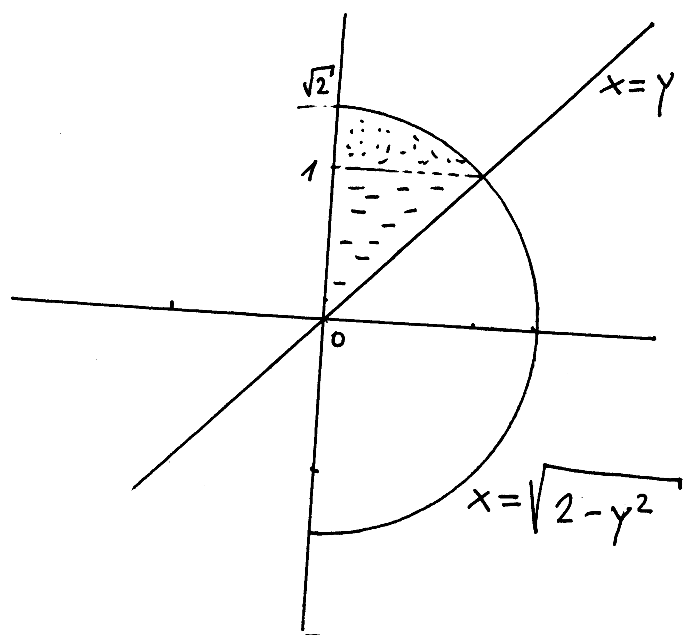
$y=\sqrt{2-x^2}$ parabola

$y^2=2-x^2$

$x^2+y^2=2$

krug sa centrom u tački $(0,0)$ poluprečnika $r=\sqrt{2}\approx 1{,}41$



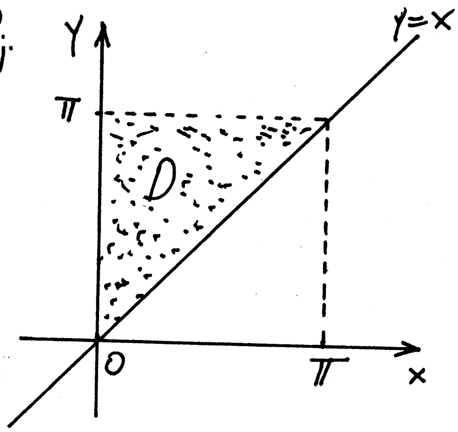


$$I=\int_0^1 dx\int_x^{\sqrt{2-x^2}} f(x,y)\,dy=\int_0^1 dy\int_0^{y} f(x,y)\,dx+\int_1^{\sqrt{2}} dy\int_0^{\sqrt{2-y^2}} f(x,y)\,dx$$

(#) Izračunati dvostruki integral $\iint_D \cos(x+y)\,dx\,dy$

ako je $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le \pi\ ;\ x\le y\le \pi\}$

Rj.



$$\iint_D \cos(x+y)\,dx\,dy = \int_0^{\pi} dx \int_x^{\pi} \cos(x+y)\,dy =$$

$$= \int_0^{\pi} dx\, \sin(x+y)\Big|_x^{\pi} = \int_0^{\pi} \left[\sin(x+\pi) - \sin 2x\right] dx \overset{(*)}{=}$$

$$\sin(x+\pi) = \sin x\cos\pi + \sin\pi\cos x = -\sin x$$

$$\overset{(*)}{=} \int_0^{\pi} (-\sin x - \sin 2x)\,dx = -\int_0^{\pi} \sin x\,dx - \int_0^{\pi} \sin 2x\,dx = \cos x\Big|_0^{\pi} + \frac{1}{2}\cos 2x\Big|_0^{\pi} =$$

$$= (-1-1) + \frac{1}{2}(1-1) = -2$$

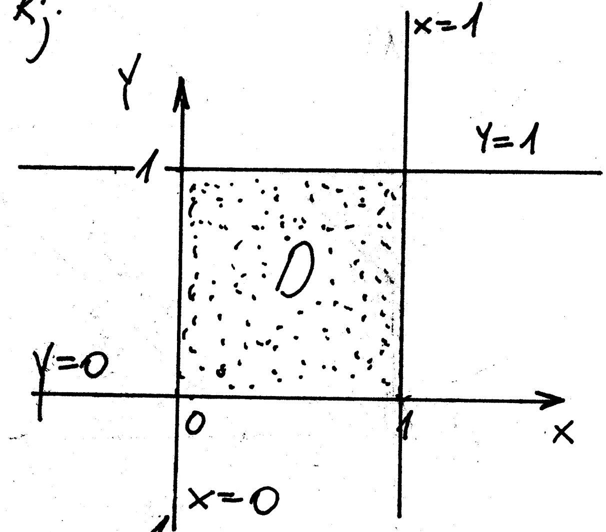
II način

$$\iint_D \cos(x+y)\,dx\,dy = \int_0^{\pi} dy \int_0^{y} \cos(x+y)\,dx = \int_0^{\pi} dy\, \sin(x+y)\Big|_0^{y} =$$

$$\int_0^{\pi} (\sin 2y - \sin y)\,dy = -\frac{1}{2}\cos 2y\Big|_0^{\pi} + \cos y\Big|_0^{\pi} = -\frac{1}{2}(1-1) + (-1-1) = -2$$

(#) Izračunati vrijednost integrala $I=\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy$ gdje je $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1;\ 0\le y\le 1\}$.

Rj.



I način

$$\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy=\int_0^1 dx\int_0^1 \frac{x^2}{1+y^2}\,dy=$$

$$=\int_0^1 x^2\,dx\int_0^1 \frac{dy}{1+y^2}=\int_0^1 x^2 \operatorname{arctg} y\Big|_0^1 dx=$$

$$=\frac{\pi}{4}\int_0^1 x^2\,dx=\frac{\pi}{4}\cdot\frac{x^3}{3}\Big|_0^1=\frac{\pi}{12}$$

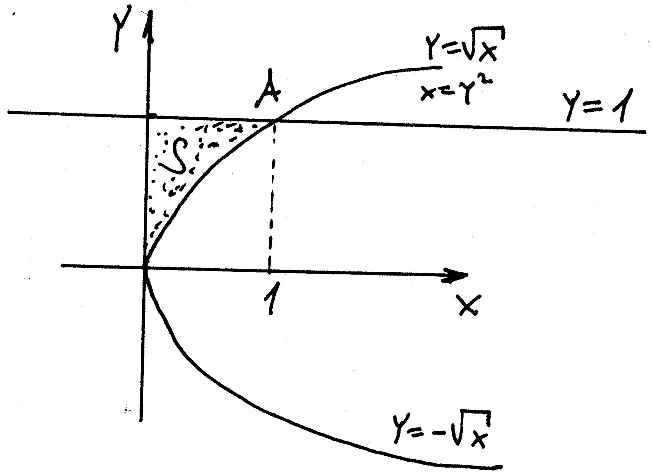
II način

$$\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy=\int_0^1 dy\int_0^1 \frac{x^2}{1+y^2}\,dx=\int_0^1 \frac{dy}{1+y^2}\int_0^1 x^2\,dx$$

$$=\int_0^1 \frac{1}{1+y^2}\cdot\frac{x^3}{3}\Big|_0^1 dy=\frac{1}{3}\int_0^1 \frac{dy}{1+y^2}=\frac{1}{3}\operatorname{arctg} x\Big|_0^1=\frac{1}{3}\cdot\frac{\pi}{4}=\frac{\pi}{12}$$

(#) Izračunati integral $\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy$, gdje je S oblast ograđena parabolom $y^2=x$, te pravama $x=0$, $y=1$.

Rj. Nacrtajmo sliku



Tačka $A(1,1)$ je presjek parabole $y^2=x$ i prave $y=1$.

Moguća su dva načina:

$$\iint\limits_S e^{-\frac{x}{y}}dx\,dy=\int_0^1\Big[\int_0^{y^2} e^{-\frac{x}{y}}dx\Big]dy$$

ili

$$\iint\limits_S e^{-\frac{x}{y}}dx\,dy=\int_0^1\Big[\int_{\sqrt{x}}^{1} e^{-\frac{x}{y}}dy\Big]dx$$

Kako $\int e^{-\frac{1}{t}}dt=?$ imamo:

$$\iint\limits_S e^{-\frac{x}{y}}dx\,dy=\int_0^1\Big[\int_0^{y^2} e^{-\frac{x}{y}}dx\Big]dy=\int_0^1\Big(-y\,e^{-\frac{x}{y}}\Big|_0^{y^2}\Big)dy=-\int_0^1 y(e^{-y}-1)\,dy \overset{(*)}{=}$$

$$\int e^{-\frac{x}{y}}dx=\left|\begin{matrix}-\frac{x}{y}=t\\ -\frac{1}{y}dx=dt\end{matrix}\right|=\int e^t(-y)\,dt=-y\,e^t+C=-y\,e^{-\frac{x}{y}}+C$$

$$\overset{(*)}{=}\int_0^1(y-ye^{-y})\,dy=\int_0^1 y\,dy+\int_0^1(-y)e^{-y}dy=\frac{1}{2}y^2\Big|_0^1+(y+1)\,e^{-y}\Big|_0^1\overset{(\Delta)}{=}$$

$$\int t\,e^t\,dt=\left|\begin{matrix}u=t & dv=e^t dt\\ du=dt & v=e^t\end{matrix}\right|=te^t-\int e^t dt=(t-1)e^t+C$$

$$\int(-t)\,e^{-t}dt=\left|\begin{matrix}u=-t & dv=e^{-t}dt\\ du=-dt & v=-e^{-t}\end{matrix}\right|=te^{-t}-\int e^{-t}dt=(t+1)e^{-t}+C$$

$$\overset{(\Delta)}{=}\frac{1}{2}+(2e^{-1}-1)=\frac{2}{e}-\frac{1}{2}$$

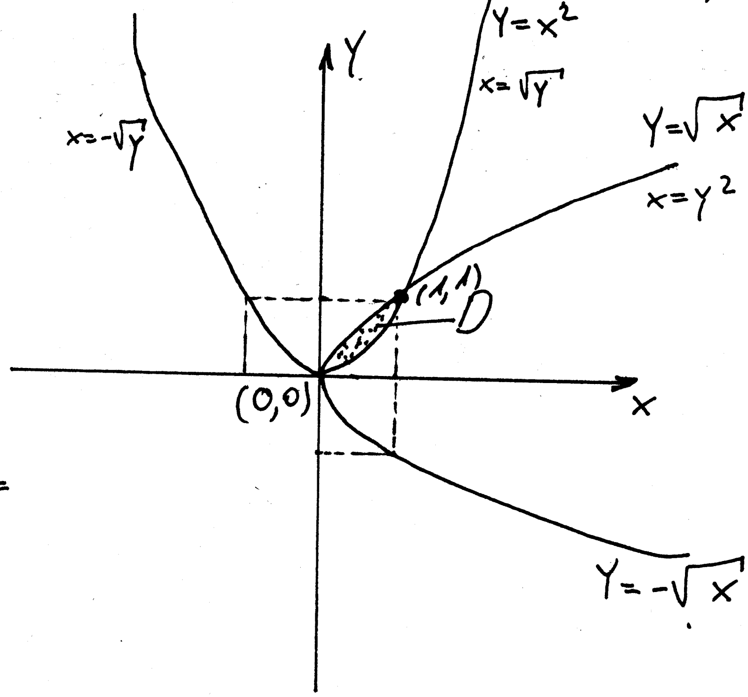
(#) Izračunati dvostruki integral $\iint\limits_D (x^2+y)\,dx\,dy$

gdje je D površ ograničena linijama $y=x^2$ i $y^2=x$.

Rj. Nađimo presječnu tačku i nacrtajmo ove dvije krive

$$y=x^2$$
$$y^2=x$$
$$x^4=x$$
$$x(x^3-1)=0$$
$$x(x-1)(x^2+x+1)=0$$
$$x=0 \text{ ili } x=1$$

Presječne tačke krivih su $(0,0)$ i $(1,1)$.



$$\iint\limits_D (x^2+y)\,dx\,dy=\int_0^1 dx\int_{x^2}^{\sqrt{x}}(x^2+y)\,dy=$$

$$=\int_0^1 dx\left(x^2y\Big|_{x^2}^{\sqrt{x}}+\frac{1}{2}y^2\Big|_{x^2}^{\sqrt{x}}\right)=\int_0^1\left[x^2(\sqrt{x}-x^2)+\frac{1}{2}(x-x^4)\right]dx$$

$$=\int_0^1\left(x^2\sqrt{x}\underline{-x^4}+\frac{1}{2}x\underline{-\frac{1}{2}x^4}\right)dx=\int_0^1\left(-\frac{3}{2}x^4+x^{\frac{5}{2}}+\frac{1}{2}x\right)dx=$$

$$=-\frac{3}{2}\cdot\frac{1}{5}x^5\Big|_0^1+\frac{2}{7}\cdot x^{\frac{7}{2}}\Big|_0^1+\frac{1}{2}\cdot\frac{1}{2}x^2\Big|_0^1=-\frac{3}{10}+\frac{2}{7}+\frac{1}{4}=\frac{-3\cdot 14+2\cdot 20+1\cdot 35}{140}$$

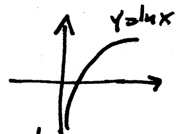
$$=\frac{-42+40+35}{140}=\frac{33}{140}$$

II način: $\iint\limits_D (x^2+y)\,dx\,dy=\int_0^1\left(\int_{y^2}^{\sqrt{y}}(x^2+y)\,dx\right)dy=\ldots=\int_0^1\left(\frac{4}{3}\sqrt{y^3}-y^3-\frac{1}{3}y^6\right)dy$

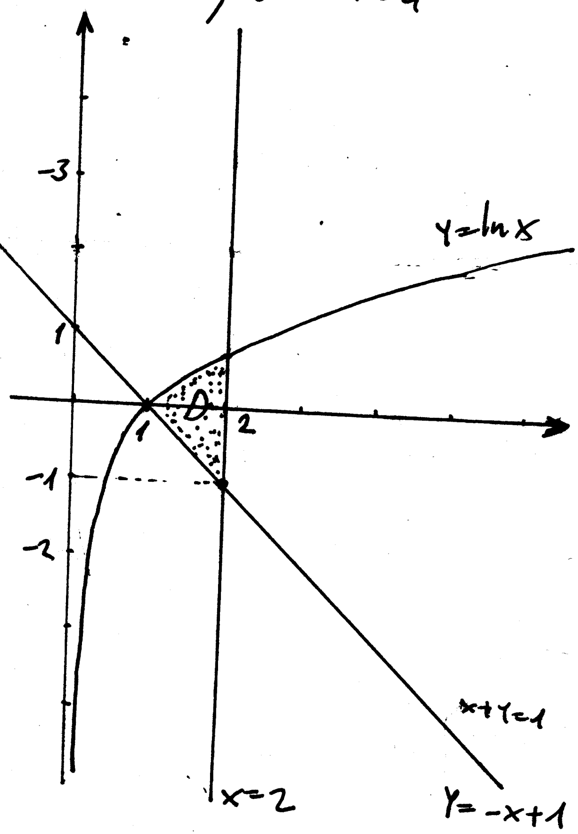
za vježbu

$$=\ldots=\frac{33}{140}$$

(#) Izračunati dvostruki integral $I=\iint\limits_D xy\,dx\,dy$, gdje je $D:\ y=\ln x,\ \ x=2,\ \ x+y=1.$

Rj. Kriva $y=\ln x$ izgleda ovako 

Nacrtajmo sliku



Pronađimo presječne tačke datih krivi.

$$\begin{array}{c} y=\ln x\\ x=2\\ \hline y=\ln 2\approx 0{,}69\\ (2,\ln 2)\end{array}\qquad
\begin{array}{c} y=\ln x\\ x+y=1\\ \hline y=\ln x\\ y=-x+1\\ \hline \ln x=-x+1\\ x=1\\ (1,0)\end{array}\qquad
\begin{array}{c} x=2\\ x+y=1\\ \hline 2+y=1\\ y=-1\\ (2,-1)\end{array}$$

$$I=\iint\limits_D xy\,dx\,dy=\int_1^2 dx\int_{-x+1}^{\ln x} xy\,dy=\int_1^2 x\,dx\int_{-x+1}^{\ln x} y\,dy=$$

$$=\int_1^2 x\left(\frac12 y^2\Big|_{-x+1}^{\ln x}\right)dx=\frac12\int_1^2 x\big(\ln^2 x-\underbrace{(-x+1)^2}_{x^2-2x+1}\big)\,dx=$$

$$=\frac12\int_1^2 x\ln^2 x\,dx-\frac12\int_1^2 (x^3-2x^2+x)\,dx$$

$$\int_1^2 x\ln^2 x\,dx=\left|\begin{array}{ll} u=\ln^2 x & dv=x\,dx\\ du=2\ln x\cdot\frac1x\,dx & v=\frac12 x^2\end{array}\right|=\overbrace{\frac12 x^2\ln^2 x\Big|_1^2}^{\overset{2\cdot 0{,}48\ \approx\ 0{,}96}{\frac12 4(\ln 2)^2}}-\int_1^2 x\ln x\,dx=$$

$$=\left|\begin{array}{ll} u=\ln x & dv=x\,dx\\ du=\frac1x dx & v=\frac12 x^2\end{array}\right|=2\ln^2 2-\Big[\overbrace{\frac12 x^2\ln x\Big|_1^2}^{2\ln 2}-\frac12\underbrace{\int_1^2 x\,dx}_{\substack{\frac12 x^2|_1^2\\ \frac12\cdot 3}}\Big]=2\ln^2 2-2\ln 2+\frac34$$

$$\int_1^2 (x^3-2x^2+x)\,dx=\frac14 x^4\Big|_1^2-\frac23 x^3\Big|_1^2+\frac12 x^2\Big|_1^2=$$

$$=\frac{15}{4}-\frac{14}{3}+\frac32=\frac{45-56+18}{12}=\frac{7}{12}$$

traženo rješenje

$$I=\frac12\left(2\ln^2 2-2\ln 2+\frac34\right)-\frac12\cdot\frac{7}{12}=\ln^2 2-\ln 2+\frac38-\frac{7}{24}=\ln^2 2-\ln 2+\frac{1}{12}$$

Izračunati dvostruki integral $I=\iint\limits_D xy\,dx\,dy$, gdje je D oblast ograničena linijama $xy=1$, $x+y=\frac{5}{2}$.

Rj. Skiciramo oblast D

$xy=1$

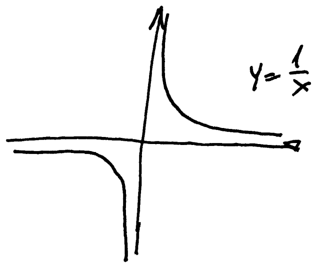
$y=\frac{1}{x}$ $\quad \mathcal{D}: x\in\mathbb{R}\setminus\{0\}$

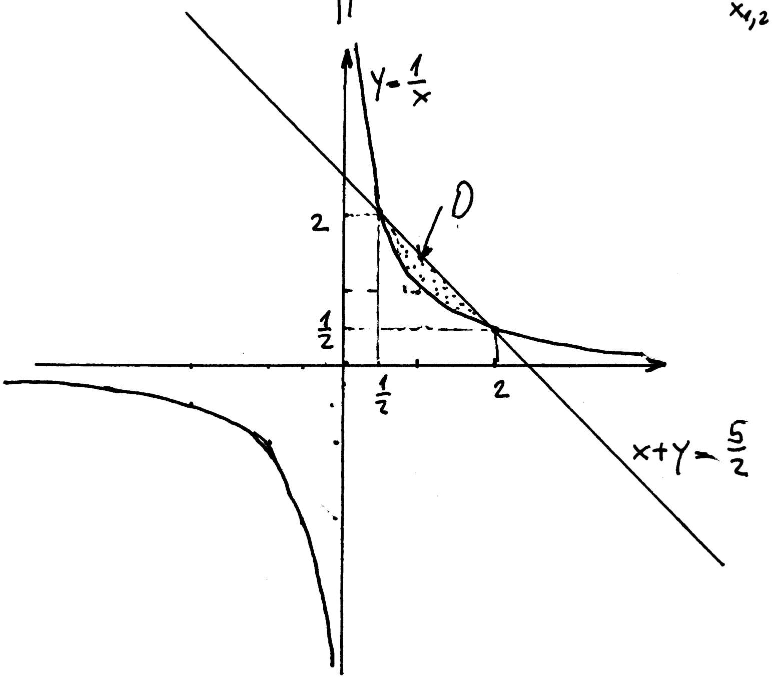
f-ja je neparna (simetrična u odnosu na 0)

ne siječe y-osu, ne siječe x-osu

$\lim\limits_{x\to0+}\frac{1}{x}=+\infty$

$\lim\limits_{x\to\infty}\frac{1}{x}=0$





Nađimo presječne tačke krive $xy=1$ i prave $x+y=\frac{5}{2}$.

$$\begin{aligned}&xy=1\\&x+y=\frac{5}{2}\\\hline&y=\frac{1}{x}\\&x+y=\frac{5}{2}\\\hline&x+\frac{1}{x}=\frac{5}{2}\quad /\cdot x\\&x^2-\frac{5}{2}x+1=0\quad /\cdot 2\\&2x^2-5x+2=0\\&D=25-16=9\\&x_{1,2}=\frac{5\pm3}{4}\qquad x_1=\frac{2}{4}=\frac{1}{2}\\&\qquad\qquad\qquad x_2=2\end{aligned}$$

$x_1=\frac{1}{2}\Rightarrow y_1=2$

$x_2=2\Rightarrow y=\frac{1}{2}$

$$D:\begin{cases}\frac{1}{2}<x<2\\ \frac{1}{x}<y<\frac{5}{2}-x\end{cases}$$

$$\iint\limits_D xy\,dx\,dy=\int_{\frac12}^{2}x\,dx\int_{\frac1x}^{\frac52-x}y\,dy=$$

$$=\int_{\frac12}^{2}x\,\frac{1}{2}y^2\Big|_{\frac1x}^{\frac52-x}dx=\frac{1}{2}\int_{\frac12}^{2}x\left(\left(\frac{5}{2}-x\right)^2-\frac{1}{x^2}\right)dx=\frac{1}{2}\int_{\frac12}^{2}x\left(\frac{25}{4}-5x+x^2-\frac{1}{x^2}\right)dx$$

$$\frac{1}{2}\int_{\frac12}^{2}\left(\frac{25}{4}x-5x^2+x^3-\frac{1}{x}\right)dx=\int_{\frac12}^{2}\left(\frac{1}{2}x^3-\frac{5}{2}x^2+\frac{25}{8}x-\frac{1}{2}\cdot\frac{1}{x}\right)dx=\ldots=\frac{165}{128}-\ln 2$$

rješenje

Promjeniti poredak integracije u integralu

$$I=\int_{-7}^{1} dy \int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x,y)\,dx$$

Rj.

Ako sa D označimo oblast integracije imamo

$$D:\begin{cases} 2-\sqrt{7-6y-y^2} \le x \le 2+\sqrt{7-6y-y^2} \\ -7 \le y \le 1 \end{cases}$$

Posmatrajmo prvo samo promjenjivu x:

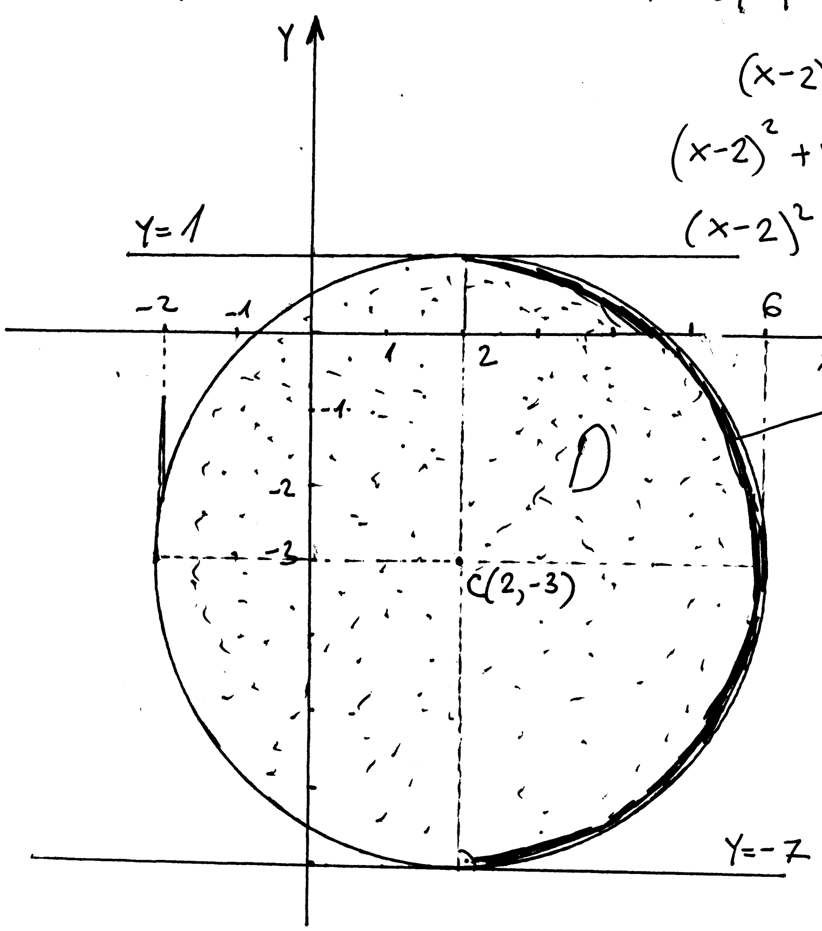
$$-\sqrt{7-6y-y^2} \le x-2 \le \sqrt{7-6y-y^2}$$

$$(x-2)^2 = 7-6y-y^2$$

$$(x-2)^2 + y^2 + 2\cdot 3\cdot y + 9 - 9 = 7$$

$$(x-2)^2 + (y+3)^2 = 16$$

krug sa centrom u tački $C(2,-3)$ poluprečnika $r=4$



$$(y+3)^2 = 16 - \underbrace{(x-2)^2}_{x^2-4x+4}$$

$$(y+3)^2 = 12 - x^2 + 4x$$

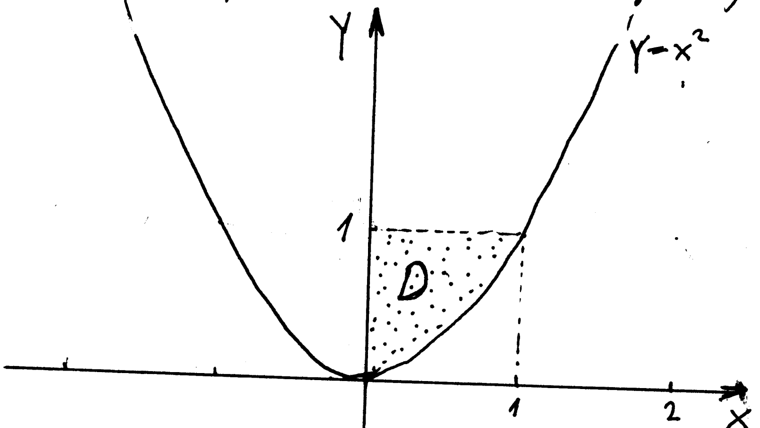
$$y_{1,2} = -3 \pm \sqrt{12-x^2+4x}$$

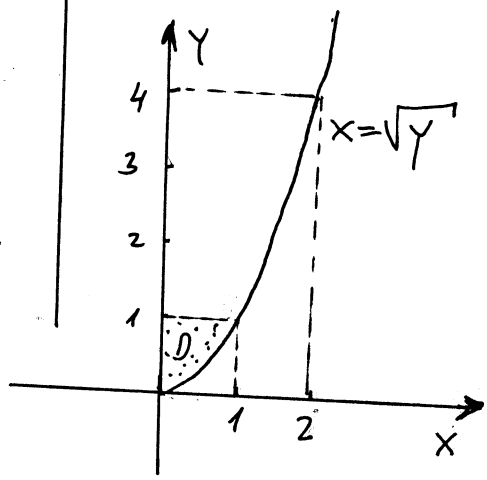
Prema tome

$$I=\int_{-7}^{1} dy \int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x,y)\,dx = \int_{-2}^{6} dx \int_{-3-\sqrt{12-x^2+4x}}^{-3+\sqrt{12-x^2+4x}} f(x,y)\,dx \,.$$

(#) Izračunati integral $I=\int_0^1 x^5 dx \int_{x^2}^1 e^{y^2} dy$.

Rj. Skicirajmo oblast integracije D





$$I=\int_0^1 x^5 dx \int_{x^2}^1 e^{y^2} dy =$$

$$=\iint_D x^5 e^{y^2} dx\, dy =$$

$$=\int_0^1 e^{y^2} dy \int_0^{\sqrt{y}} x^5 dx =$$

$$=\int_0^1 e^{y^2} \frac{1}{6} x^6 \Big|_0^{\sqrt{y}} dy =$$

$$=\frac{1}{6}\int_0^1 e^{y^2} y^3 dy = \left| \begin{array}{ll} u=y^2 & dv=e^{y^2} y\, dy = \frac{1}{2} e^{y^2} d(y^2) \\ du=2y\,dy & v=\frac{1}{2} e^{y^2} \end{array} \right| =$$

$$=\frac{1}{12} y^2 e^{y^2}\Big|_0^1 - \frac{1}{6}\int_0^1 e^{y^2} y\, dy = \frac{1}{12}(1\cdot e^1 - 0) - \frac{1}{12}\int_0^1 e^{y^2} d(y^2) =$$

$$=\frac{1}{12}e - \frac{1}{12} e^{y^2}\Big|_0^1 = \frac{1}{12}e - \frac{1}{12}(e-1) = \frac{1}{12}e - \frac{1}{12}e + \frac{1}{12} = \frac{1}{12}$$

traženo rješenje

Izračunati vrijednost integrala $I=\iint\limits_{|x|+|y|\le 1} x^2\,dx\,dy$

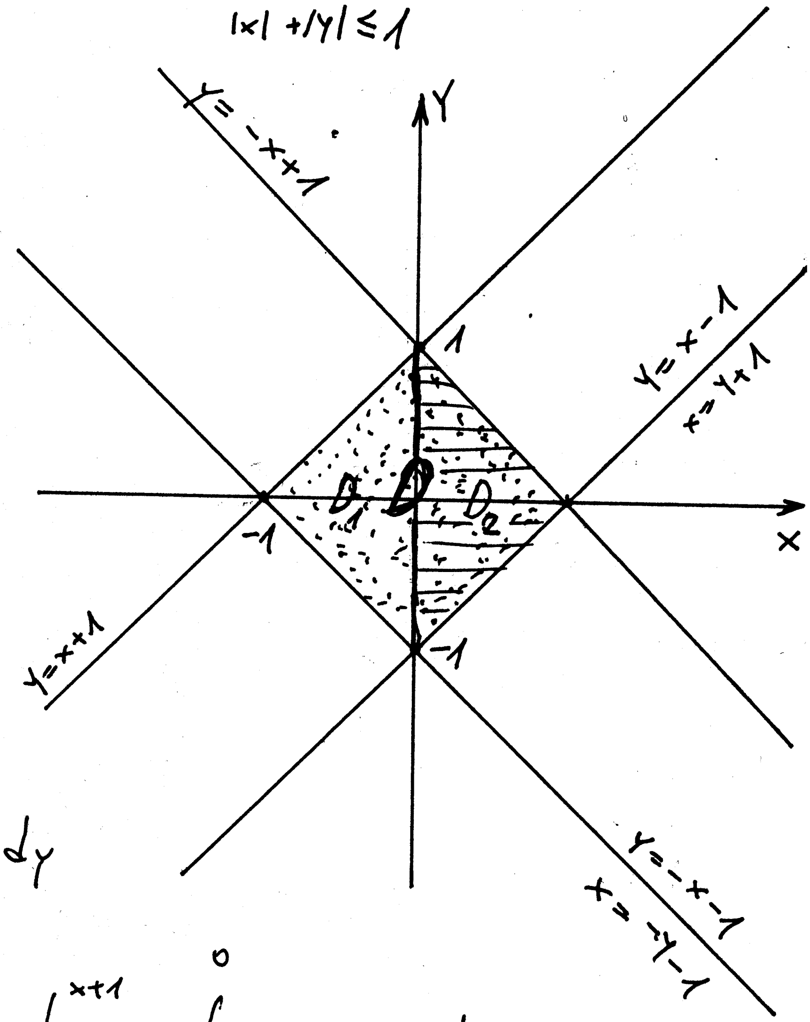
Rj:

$x<0,\ y<0 \Rightarrow -x-y\le 1$
$\qquad y\ge -x-1$

$x<0,\ y>0 \Rightarrow -x+y\le 1$
$\qquad y\le x+1$

$x\ge 0,\ y<0 \Rightarrow x-y\le 1$
$\qquad y\ge x-1$

$x\ge 0,\ y\ge 0 \Rightarrow x+y\le 1$
$\qquad y\le -x+1$



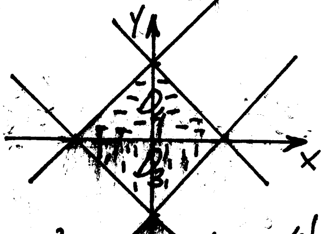
$$I=\iint\limits_{|x|+|y|\le 1} x^2\,dx\,dy=\iint\limits_{D_1} x^2\,dx\,dy+\iint\limits_{D_2} x^2\,dx\,dy$$

$$\iint\limits_{D_1} x^2\,dx\,dy=\int_{-1}^{0}dx\int_{-x-1}^{x+1}x^2\,dy=\int_{-1}^{0}x^2\,dx\ \underbrace{y\Big|_{-x-1}^{x+1}}_{x+1+x+1}=\int_{-1}^{0}x^2(2x+2)\,dx=$$

$$=\int_{-1}^{0}(2x^3+2x^2)\,dx=\frac{2}{4}x^4\Big|_{-1}^{0}+\frac{2}{3}x^3\Big|_{-1}^{0}=\frac{1}{2}\cdot(-1)+\frac{2}{3}\cdot 1=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$$

$$\iint\limits_{D_2} x^2\,dx\,dy=\int_{0}^{1}dx\int_{x-1}^{-x+1}x^2\,dy=\int_{0}^{1}x^2\,dx\ \underbrace{y\Big|_{x-1}^{-x+1}}_{-x+1-x+1}=\int_{0}^{1}x^2(-2x+2)\,dx=$$

$$\int_{0}^{1}(-2x^3+2x^2)\,dx=-\frac{2}{4}x^4\Big|_{0}^{1}+\frac{2}{3}x^3\Big|_{0}^{1}=-\frac{1}{2}+\frac{2}{3}=\frac{1}{6},\qquad I=\iint\limits_{|x|+|y|\le 1}x^2\,dx\,dy=\frac{2}{6}$$



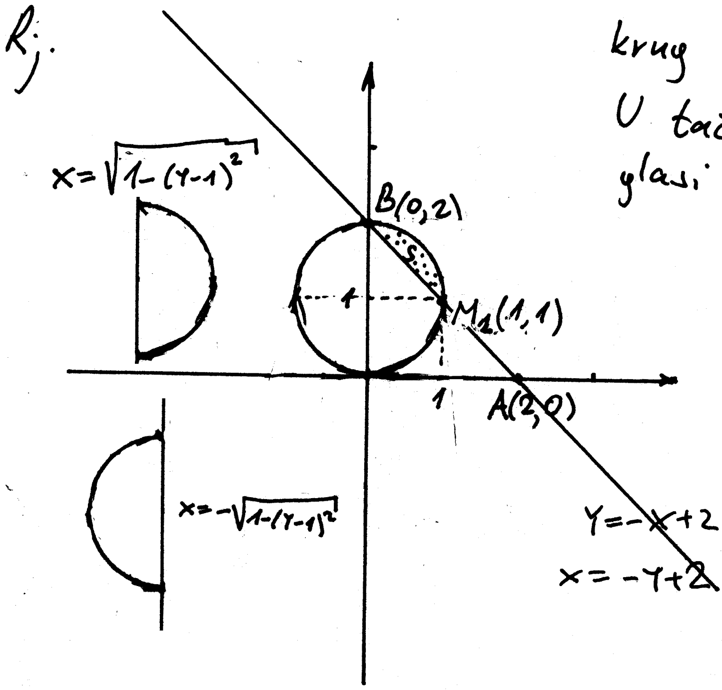
II način: $I=\iint\limits_{D_3}x^2\,dx\,dy+\iint\limits_{D_4}x^2\,dx\,dy$

$$\iint\limits_{D_3}x^2\,dx\,dy=\int_{-1}^{0}dy\int_{-y-1}^{y+1}x^2\,dx=\int_{-1}^{0}dy\,\frac{1}{3}\,\underbrace{x^3\Big|_{-y-1}^{y+1}}_{(-1)(y+1)}=\frac{1}{3}\int_{-1}^{0}2(y+1)^3\,dy=\frac{2}{3}\int_{-1}^{0}(y+1)^3\,dy=\frac{2}{3}\cdot\frac{1}{4}(y+1)^4\Big|_{-1}^{0}$$

$$=\frac{1}{6}\ \ldots$$

(#) Izračunati dvostruki integral $\iint\limits_S x\,dx\,dy$ gdje je područje integracije S ograničeno pravcem koji prolazi tačkama $A(2,0)$, $B(0,2)$ i lukom kruga poluprečnika 1 sa centrom u tački $C(0,1)$.

Rj.



$\overset{x_1,y_1}{A(2,0)}$, $\overset{x_2,y_2}{B(0,2)}$

krug $x^2+(y-1)^2=1$

U tačkama $A(2,0)$ i $B(0,2)$ jednačina prave glasi

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-0=\frac{2-0}{0-2}(x-2)$$

$$y=-x+2$$

Nađimo presječne tačke prave i kruga

$$x^2+(y-1)^2=1$$
$$y=-x+2$$
$$x^2+(-x+2-1)^2=1$$

$$x^2+(-x+1)^2=1$$
$$x^2+x^2-2x+1=1$$
$$2x^2-2x=0$$
$$2x(x-1)=0$$

$x_1=0$
$x_2=1$

$x_1=0 \Rightarrow y=2$
$x_2=1 \Rightarrow y=1$

Presječne tačke prave i kruga su $M_1(0,2)$ i $M_2(1,1)$

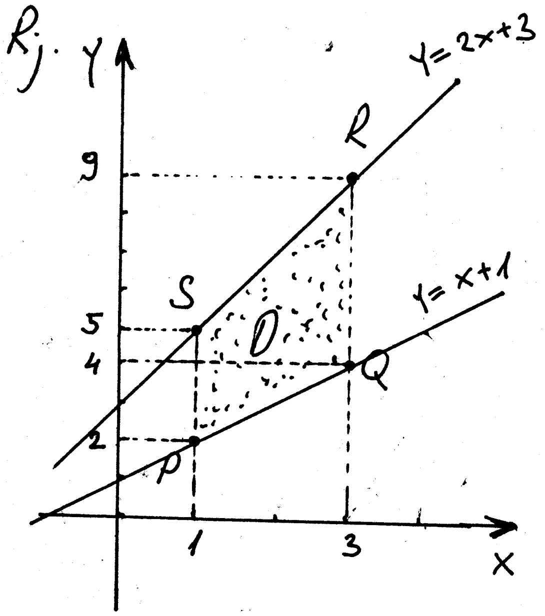
$$\iint\limits_S x\,dx\,dy=\int_1^2\left[\int_{-y+2}^{\sqrt{1-(y-1)^2}} x\,dx\right]dy=\int_1^2 \frac{1}{2}x^2\Big|_{-y+2}^{\sqrt{1-(y-1)^2}}dy=\frac{1}{2}\int_1^2\left[(1-(y-1)^2)-(2-y)^2\right]dy$$

$$=\frac{1}{2}\int_1^2\left[1-y^2+2y-1-4+4y-y^2\right]dy=\frac{1}{2}\int_1^2(-2y^2+6y-4)\,dy=\frac{1}{2}\cdot 2\int_1^2(-y^2+3y-2)\,dy$$

$$=-\frac{1}{3}y^3\Big|_1^2+\frac{3}{2}y^2\Big|_1^2-2y\Big|_1^2=-\frac{7}{3}+\frac{9}{2}-2=\frac{-14+27-12}{6}=\frac{1}{6}$$

Izračunati $\iint_D x\,dx\,dy$ pri čemu je D četverougao $\square PQRS$ gdje su tačke $P(1,2)$, $Q(3,4)$, $R(3,9)$ i $S(1,5)$.

Rj.



$$\iint_D x\,dx\,dy = \int_1^3\left[\int_{x+1}^{2x+3} x\,dy\right]dx =$$

$y-y_1=k(x-x_1)$

$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$P(\overset{x_1}{1},\overset{y_1}{2})$

$Q(3,4)$

$y-2=\frac{2}{2}(x-1)$

$y=x+1$

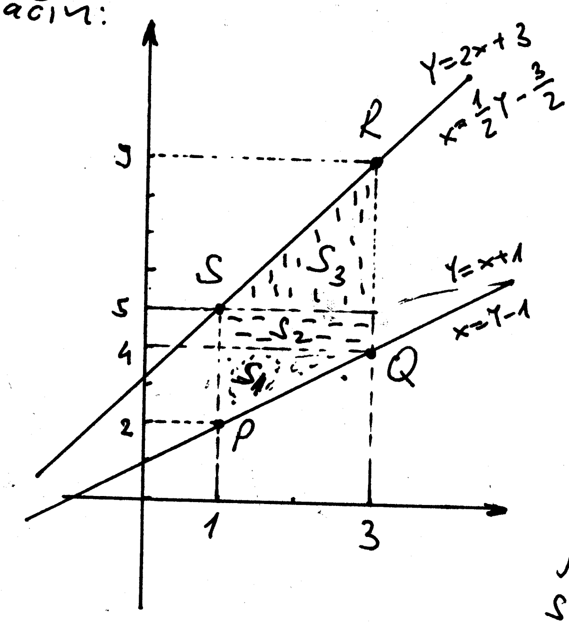
$R(\overset{x_1}{3},\overset{y_1}{9})$

$S(1,5)$ $\quad y-9=\frac{-4}{-2}(x-3) \Rightarrow y=2x+3$

$$=\int_1^3 x\,y\Big|_{x+1}^{2x+3}dx=\int_1^3 x(2x+3-x-1)\,dx=\int_1^3(x^2+2x)\,dx=\frac{1}{3}x^3\Big|_1^3+x^2\Big|_1^3=$$

$$=\frac{1}{3}(27-1)+(9-1)=\frac{26}{3}+8=\frac{50}{3}$$

II način: $\quad S:\ \square PQRS$



$$\iint_S x\,dx\,dy=\iint_{S_1}x\,dx\,dy+\iint_{S_2}x\,dx\,dy+\iint_{S_3}x\,dx\,dy$$

$$\iint_{S_1}x\,dx\,dy=\int_2^4\left[\int_1^{y-1}x\,dx\right]dy=\dots=\int_2^4\left(\frac{1}{2}y^2-y\right)dy=\dots=\frac{10}{3}$$

$$\iint_{S_2}x\,dx\,dy=\int_4^5\left[\int_1^3 x\,dx\right]dy=\dots=\int_4^5 4\,dy=\dots=4$$

$$\iint_{S_3}x\,dx\,dy=\int_5^9\left[\int_{\frac{1}{2}y-\frac{3}{2}}^{3}x\,dx\right]dy=\dots=\int_5^9\left(\frac{9}{2}-\frac{(y-3)^2}{8}\right)dy=\dots=\frac{28}{3}$$

#) Izračunati $\iint\limits_D (x+y)\,dx\,dy$ ako je D oblast ograničena linijama $y^2=2x$, $x+y=4$ i $x+y=12$.

Rj. Nađimo presječne tačke ovih linija.

$$\begin{array}{l} y^2=2x \\ \underline{x+y=4} \\ y^2=2x \\ \underline{y=4-x} \end{array}$$

$(4-x)^2=2x$

$16-8x+x^2=2x$

$x^2-10x+16=0$

$D=100-64=36$

$x_{1,2}=\frac{10\pm 6}{2}$ $\quad x_1=2$, $x_2=8$

$A(2,2)$

$B(8,-4)$

$$\begin{array}{l} y^2=2x \\ \underline{x+y=12} \\ y^2=2x \\ \underline{y=12-x} \end{array}$$

$(12-x)^2=2x$

$144-24x+x^2=2x$

$x^2-26x+144=0$

$D=676-576$

$D=100$

$x_{1,2}=\frac{26\pm 10}{2}$

$x_1=8$ $\quad x_2=18$

$C(8,4)$

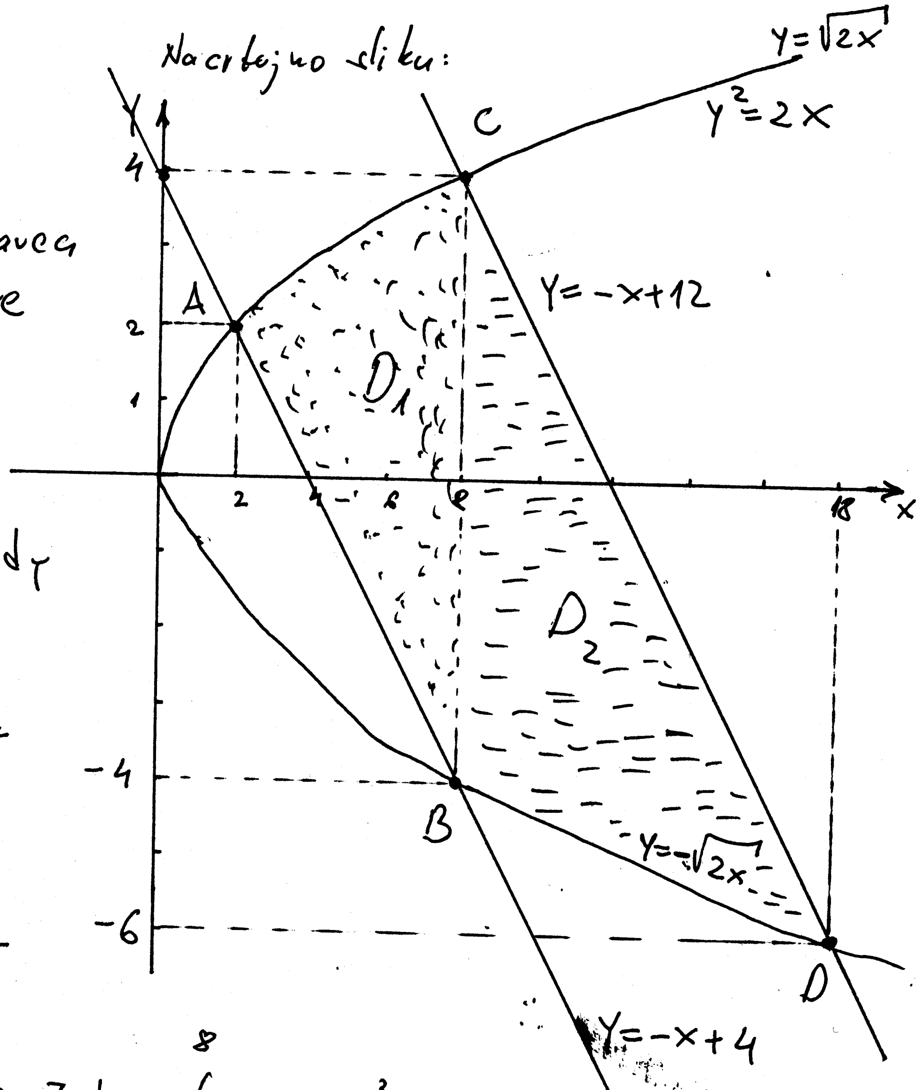
$D(18,-6)$

$$\begin{array}{l} x+y=4 \\ \underline{x+y=12} \\ y=-x+4 \\ y=-x+12 \end{array}$$

ove dvije prave imaju isti koeficijent pravca

dvije paralelne prave

Nacrtajmo sliku:



$$I=\iint\limits_D (x+y)\,dx\,dy=\iint\limits_{D_1}(x+y)\,dx\,dy+\iint\limits_{D_2}(x+y)\,dx\,dy$$

$$\iint\limits_{D_1}(x+y)\,dx\,dy=\int_2^8\left[\int_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}(x+y)\,dy\right]dx=$$

$$=\int_2^8\left(xy\Big|_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}+\frac{1}{2}y^2\Big|_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}\right)dx=$$

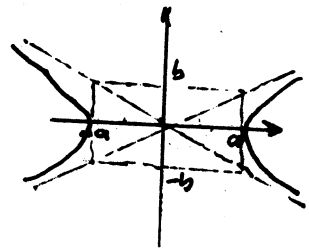
$$=\int_2^8\left[x\left(\sqrt{2}\sqrt{x}+x-4\right)+\frac{1}{2}\left(2x-(-x+4)^2\right)\right]dx=\ldots=\int_2^8\left(x+\sqrt{2}\,x^{\frac{3}{2}}+\frac{1}{2}x^2-8\right)dx=\ldots=\frac{826}{5}$$

$$\iint\limits_{D_2}(x+y)\,dx\,dy=\int_8^{18}\left[\int_{-\sqrt{2x}}^{-x+12}(x+y)\,dy\right]dx=\ldots=\int_8^{18}\left(\sqrt{2}\,x^{\frac{3}{2}}-x-\frac{1}{2}x+72\right)dx=\ldots=\frac{5678}{15}$$

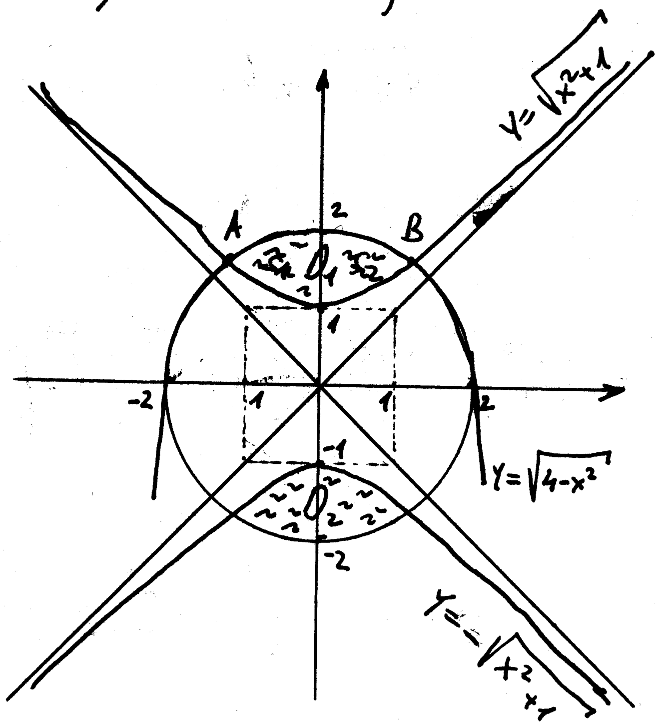
$I=543\frac{11}{15}$ vrijedn. dvostr. integr.

Izračunati $I=\iint_D dx\,dy$, ako je $D: y^2-x^2=1,\ x^2+y^2=4$.

Rj. Krive oblika $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ zovemo hiperbole i one su oblika



Skicirajmo naše dvije krive



$x^2+y^2=4$ je krug sa centrom u $(0,0)$ poluprečnika $r=2$

$$D=D_1\cup D_2$$

$$I=\iint_D dx\,dy=\iint_{D_1\cup D_2} dx\,dy=2\iint_{D_1} dx\,dy$$

$y^2=4-x^2$ $\qquad$ $y^2=x^2+1$

$y=\pm\sqrt{4-x^2}$ $\qquad$ $y=\pm\sqrt{x^2+1}$

Nađimo presečnu tačku krivih $y=\sqrt{x^2+1}$ i $y=\sqrt{4-x^2}$

$$\sqrt{x^2+1}=\sqrt{4-x^2} \quad /^2$$

$$x^2+1=4-x^2$$

$$2x^2-3=0$$

$$2x^2=3$$

$$x^2=\frac{3}{2}$$

$x_1=-\sqrt{\frac{3}{2}}$

$x_2=\sqrt{\frac{3}{2}}$

$x_1=-\sqrt{\frac{3}{2}} \Rightarrow y=\sqrt{\frac{3}{2}+1}=\sqrt{\frac{5}{2}}$

Presečne tačke su $A\left(-\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$ i $B\left(\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$.

Primetimo da je oblast D_1 simetrična

$$\iint_{D_1} dx\,dy=\iint_{S_1} dx\,dy+\iint_{S_2} dx\,dy=2\iint_{S_2} dx\,dy=2\int_0^{\sqrt{\frac{3}{2}}} dx\int_{\sqrt{x^2+1}}^{\sqrt{4-x^2}} dy=2\int_0^{\sqrt{\frac{3}{2}}}\left(\sqrt{4-x^2}-\sqrt{x^2+1}\right)dx$$

$$\int\sqrt{4-x^2}\,dx=2\arcsin\frac{x}{2}+\frac{x\sqrt{4-x^2}}{2}$$ (sami)

(metodom Ostrogradskog)

$$\int\sqrt{x^2+1}\,dx=\int\frac{x^2+1}{\sqrt{x^2+1}}\,dx=(ax+b)\sqrt{x^2+1}+\lambda\int\frac{dx}{\sqrt{x^2+1}} \quad /'$$

$$\frac{x^2+1}{\sqrt{x^2+1}} = a\sqrt{x^2+1} + (ax+b)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} + \lambda\frac{1}{\sqrt{x^2+1}} \qquad /\cdot\sqrt{x^2+1}$$

$$x^2+1 = \underline{a}(x^2+1) + \underline{a}x^2 + \underline{\underline{b}}x + \lambda$$

$$2a = 1 \Rightarrow a = \frac{1}{2}$$

$$b = 0$$

$$a + \lambda = 1 \qquad \lambda = \frac{1}{2}$$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\int\frac{dx}{\sqrt{x^2+1}}$$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|x+\sqrt{x^2+1}| + C$$

$$\int_0^{\sqrt{\frac{3}{2}}}\sqrt{4-x^2}\,dx = 2\arcsin\frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} \quad \text{(sami)}$$

$$\int_0^{\sqrt{\frac{3}{2}}}\sqrt{x^2+1}\,dx = \frac{1}{2}\cdot\sqrt{\frac{3}{2}}\,\overbrace{\sqrt{\frac{3}{2}+1}}^{\sqrt{\frac{5}{2}}} + \frac{1}{2}\ln\left|\sqrt{\frac{3}{2}}+\sqrt{\frac{3}{2}+1}\right| = \frac{\sqrt{15}}{4} + \frac{1}{2}\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

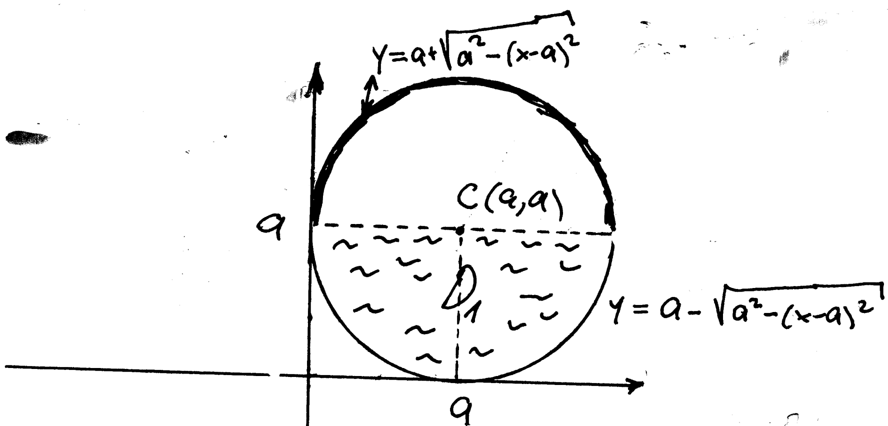
$$I = \iint_D dx\,dy = 2\iint_{D_1} dx\,dy = 2\cdot 2\iint_{S_2} dx\,dy = 4\left(2\arcsin\frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} - \frac{\sqrt{15}}{4} - \frac{1}{2}\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|\right) = 8\arcsin\frac{\sqrt{6}}{4} - 2\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

traženo rješenje.

U rješenju sljedećeg zadatka ima greška. Pronađi grešku.

(#) Izračunati dvostruki integral $\iint\limits_S \frac{dx\,dy}{\sqrt{2a-x}}$ gdje je S krug poluprečnika a, koji dodiruje koordinatne ose i leži u prvom kvadrantu.

Rj.



krug $(x-a)^2+(y-a)^2=a^2$

$$y-a=\pm\sqrt{a^2-(x-a)^2}$$

$$y=a\pm\sqrt{a^2-(x-a)^2}$$

$$\iint\limits_S \frac{dx\,dy}{\sqrt{2a-x}} = 2D_1$$

$$D_1=\int_0^{2a}\left[\int_{a-\sqrt{a^2-(x-a)^2}}^{a}\frac{1}{\sqrt{2a-x}}\,dy\right]dx =$$

$$=\int_0^{2a}\frac{dx}{\sqrt{2a-x}}\cdot y\Big|_{a-\sqrt{a^2-(x-a)^2}}^{a} = \int_0^{2a}\sqrt{\frac{a^2-(x^2-2ax+a^2)}{2a-x}}\,dx = \int_0^{2a}\sqrt{\frac{a^2-x^2+2ax-a^2}{2a-x}}\,dx$$

$$=\int_0^{2a}\sqrt{\frac{x(2a-x)}{2a-x}}\,dx = \int_0^{2a}\sqrt{x}\,dx = \int_0^{2a}x^{\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^{2a} = \frac{2}{3}\cdot(2a)^{\frac{3}{2}} =$$

$$=\frac{2}{3}\sqrt{8a^3}=\frac{2}{3}\,2a\sqrt{2a}=\frac{4a}{3}\sqrt{2a}$$ Prema tome $$\iint\limits_S\frac{dx\,dy}{\sqrt{2a-x}}=\frac{8a}{3}\sqrt{2a}$$

#

Odrediti projekcije l linije L na ravan xoy:

77. $L: 4-x^2-y^2=z, \quad z=y^2.$

78. $L: x^2+y^2=z^2, \quad (z>0), \quad x+y+z=1.$

79. $L: x^2+y^2+z^2=4, \quad x^2+y^2=z^2, \quad (z>0).$

80. $L: x^2+y^2+z^2=a^2, \quad x+y+z=0.$

81. $L: 2x+y+z=1, \quad x-y-3z=5.$

82. $L: z=x^2+y^2, \quad z=2x+2y.$

Rješenja:

77. Linija L je data kao presjek površi $z=4-x^2-y^2$ i $z=y^2$ (paraboloid i cilindar). Projekcija linije L na ravan Oxy je skup onih tačaka (x, y) za koje je aplikata z sa jedne površi jednaka aplikati z sa druge površi, dakle, taj skup određujemo iz uslova

$$4-x^2-y^2=y^2.$$

Projekcija je, dakle, elipsa

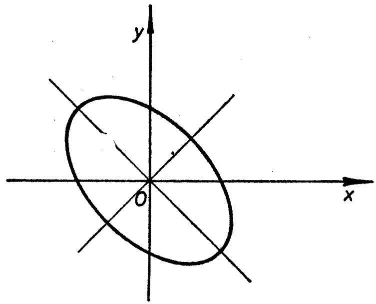
$$\frac{x^2}{2^2}+\frac{y^2}{(\sqrt{2})^2}=1.$$

78. $1=2x+2y-2xy.$

79. $x^2+y^2=2.$

80. Kriva ima projekciju

$$2x^2+2y^2+2xy-a^2=0.$$



Sl. 10

To je elipsa sa centrom u $(0, 0)$. Ose elipse su prave $y=\pm x$, a poluose su a i $\frac{a}{3}$ (sl. 10).

81. $7x+2y-8=0.$ **82.** $(x-1)^2+(y-1)^2=2.$

#

Po definiciji izračunati integrale:

83. $\iint\limits_{\substack{0\leqslant x\leqslant a\\0\leqslant y\leqslant b}} xydxdy.$ **84.** $\iint\limits_{\substack{0\leqslant x\leqslant 1\\0\leqslant y\leqslant 1}} x^2ydxdy.$

Rješenja:

83. Funkcija $f(x, y)=xy$ je neprekidna pa i integrabilna na pravougaoniku $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$. Podijelimo dati pravougaonik pravama $x=x_i$ $(i=1, \ldots, n)$, $y=y_j$ $(j=1, \ldots, m)$. Po definiciji je

$$\iint\limits_D xydxdy=\lim_{\substack{m\to\infty\\n\to\infty}}\sum_{i, j=1}^{n, m} f(M_{i,j})\cdot(x_i-x_{i-1})(y_j-y_{j-1})$$

pri čemu maksimalni podjeljak teži nuli kada $m\to\infty$, $n\to\infty$. Izaberimo da je:

$$x_i=\frac{a}{n}\cdot i,\qquad y_j=\frac{b}{m}j,$$

i da je

$$M_{i,j}=\left(\frac{a}{n}(i-1),\quad \frac{b}{m}(j-1)\right).$$

Biće:

$$\iint\limits_{\substack{0\leqslant x\leqslant a\\0\leqslant y\leqslant b}} xydxdy=\lim_{\substack{n\to\infty\\m\to\infty}}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{a}{n}(i-1)\cdot\frac{b}{m}(j-1)\cdot\frac{ab}{n\cdot m}=$$

$$=\lim_{\substack{n\to\infty\\m\to\infty}}\frac{a^2}{n^2}\cdot\frac{b^2}{m^2}\sum_{i=1}^{n}(i-1)\sum_{j=1}^{m}(j-1)=$$

$$=\lim_{\substack{n\to\infty\\m\to\infty}}\frac{a^2b^2}{n^2\cdot m^2}\cdot\frac{(n-1)\cdot n}{2}\cdot\frac{(m-1)\cdot m}{2}=\frac{a^2b^2}{4}.$$

84. $\frac{1}{6}$.

Po definiciji izračunati integral:

$$\textbf{85.}\ \iint\limits_{\substack{a\leqslant x\leqslant b\\ c\leqslant y\leqslant d}} e^{x+y}\,dxdy.$$

Rješenja:

85. Integral postoji jer je funkcija $f(x, y)=e^{x+y}$ neprekidna. Segment $[a, b]$ podijelimo pravama $x_i=a+\frac{b-a}{n}\cdot i$, $(i=0, 1, \dots, n)$, a $[c, d]$ pravama $y_j=c+\frac{d-c}{m}\cdot j$, $(j=0, 1, \dots, m)$, i u pravougaoniku $[x_{i-1}, x_i; y_{j-1}, y_j]$ uočimo tačku $M_{ij}=\left(a+\frac{b-a}{n}\cdot i,\ c+\frac{d-c}{m}\cdot j\right)$. Formirajmo integralnu sumu

$$S_{m,n}=\sum_{i,j=1}^{n,m} f(M_{i,j})(x_i-x_{i-1})(y_i-y_{i-1})=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m} e^{a+\frac{b-a}{n}\cdot i}\cdot e^{c+\frac{d-c}{m}j}\cdot\frac{b-a}{n}\cdot\frac{d-c}{m}=$$

$$=e^a\cdot e^c\cdot\frac{b-a}{n}\cdot\frac{d-c}{m}\cdot\sum_{i=1}^{n} e^{\frac{b-a}{n}\cdot i}\cdot\sum_{j=1}^{m} e^{\frac{d-c}{m}j}.$$

Dobili smo geometrijske sume, pa je

$$S_{m,n}=e^a\cdot\frac{b-a}{n}\cdot e^{\frac{b-a}{n}}\cdot\frac{1-\left(e^{\frac{b-a}{n}}\right)^n}{1-e^{\frac{b-a}{n}}}\cdot e^c\cdot\frac{d-c}{m}\cdot e^{\frac{d-c}{m}}\cdot\frac{1-\left(e^{\frac{d-c}{m}}\right)^m}{1-e^{\frac{d-c}{m}}}=$$

$$=e^a\cdot\frac{b-a}{n}\cdot e^{\frac{b-a}{n}}\cdot\frac{1-e^{b-a}}{1-e^{\frac{b-a}{n}}}\cdot e^c\cdot\frac{d-c}{n}\cdot e^{\frac{d-c}{n}}\cdot\frac{1-e^{d-c}}{1-e^{\frac{d-c}{m}}}.$$

Kako je

$$\lim_{n\to\infty}\frac{b-a}{n}\cdot\frac{1}{1-e^{\frac{b-a}{n}}}=\lim_{\alpha\to 0}\frac{\alpha}{1-e^\alpha}=-1$$

i

$$\lim_{m\to\infty}\frac{d-c}{m}\,\frac{1}{1-e^{\frac{d-c}{m}}}=-1,$$

to

$$\lim_{\substack{m\to\infty\\ n\to\infty}} S_{m,n}=e^a\cdot\frac{(1-e^{b-a})}{-1}\cdot e^c\cdot\frac{(1-e^{d-c})}{-1}=$$

$$=(e^a-e^b)\cdot(e^c-e^d).$$

Dakle,

$$\iint\limits_{\substack{a\leqslant x\leqslant b\\ c\leqslant y\leqslant d}} e^{x+y}\,dxdy=(e^a-e^b)(e^c-e^d).$$

Promijeniti poredak integracije u sljedećim integralima uzimajući da je $f(x, y)$ neprekidna funkcija:

86. $\int_0^1 dx \int_0^x f(x, y)\, dy.$

87. $\int_1^e dx \int_0^{\ln x} f(x, y)\, dy.$

88. $\int_0^4 dx \int_{3x^2}^{12x} f(x, y)\, dy.$

89. $\int_0^1 dx \int_x^{2-x} f(x, y)\, dy.$

90. $\int_0^1 dy \int_{\sqrt{y}}^{2-y} f(x, y)\, dx.$

91. $\int_{-7}^1 dx \int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x, y)\, dx.$

Rješenja:

86. $\int_0^1 dy \int_y^1 f(x, y)\, dx.$

87. $\int_0^1 dy \int_{e^y}^e f(x, y)\, dx.$

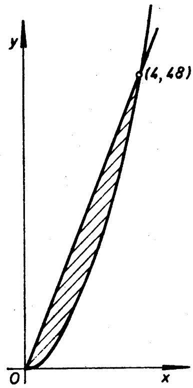
88. $x = 12x \Rightarrow x = \dfrac{y}{12},$

$$y = 3x^2 \Rightarrow x = \sqrt{\frac{y}{3}},$$

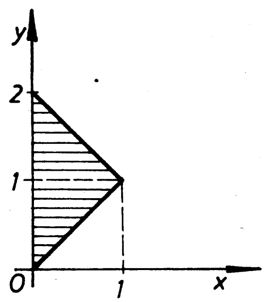
$$\int_0^{48} dy \int_{\frac{y}{12}}^{\sqrt{\frac{y}{3}}} f(x, y)\, dx \quad \text{(sl. 11).}$$

89. $\int_0^1 dy \int_0^y f(x, y)\, dx +$

$$+ \int_1^2 dy \int_0^{2-y} f(x, y)\, dx. \quad \text{(sl. 12).}$$



Sl. 11



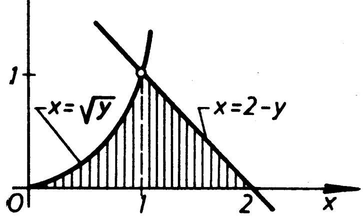
Sl. 12

90. Oblast integracije (sl. 13) dijelimo na dvije; biće

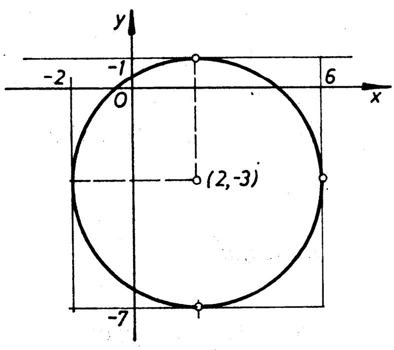
$$\int_1^2 dx \int_0^{x^2} f(x, y)\, dy + \int_1^2 dx \int_0^{2-x} f(x, y)\, dy.$$

91. Nacrtajmo linije koje ograničavaju oblast integracije $x_1 = 2 - \sqrt{7 - 6y - y^2}$, $x_2 = 2 + \sqrt{7 - 6y - y^2}$, $y = -7$, $y = 1$. Linije x_1 i x_2 su polukružnice kružnice $(x-2)^2 + (y+3)^2 = 16$. Oblast integracije je unutrašnjost kruga (sl. 14), pa je dati integral jednak integralu

$$\int_{-2}^6 dx \int_{-3-\sqrt{12-x^2+2x}}^{-3+\sqrt{12-x^2+2x}} f(x, y)\, dy.$$



Sl. 13



Sl. 14

Promijeniti poredak integracije u sljedećim integralima uzimajući da je $f(x, y)$ neprekidna funkcija:

92. $\int_0^1 dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x, y)\, dx.$ **93.** $\int_0^{\pi} dx \int_0^{\sin x} f(x, y)\, dy.$

94. $\int_{-1}^{0} dy \int_{-2\sqrt{1+y}}^{2\sqrt{1+y}} f(x, y)\, dx + \int_0^8 dy \int_{-2\sqrt{1+y}}^{2-y} f(x, y)\, dx.$

95. $\int_0^1 dx \int_{\frac{1}{2}(1-x^2)}^{\sqrt{1-x^2}} f(x, y)\, dy.$ **96.** $\int_0^1 dy \int_{\frac{y^2}{2}}^{\sqrt{3-y^2}} f(x, y)\, dx.$

Rješenja:

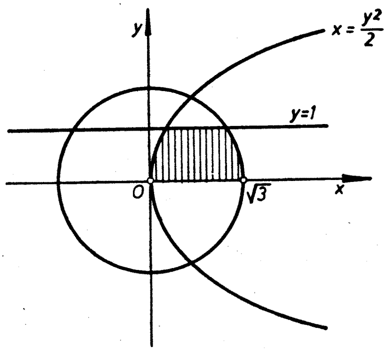
92. $\int_1^2 dx \int_{2-x}^{\sqrt{2x-x^2}} f(x, y)\, dy.$

93. $\int_0^1 dy \int_{\arcsin y}^{\pi-\arcsin y} f(x, y)\, dx.$ **94.** $\int_{-6}^{2} dx \int_{\frac{x^2}{4}-1}^{2-x} f(x, y)\, dy.$

95. $\int_0^{1/2} dy \int_{\sqrt{1-2y}}^{\sqrt{1-y^2}} f(x, y)\, dx + \int_{1/2}^{1} dy \int_0^{\sqrt{1-y^2}} f(x, y)\, dx.$

96. Oblast integracije (sl. 15) ograničena je pravama $y=0$, $y=1$, lukom parabole $x=\frac{y^2}{2}$ i lukom kružnice $x^2+y^2=3$, $(x>0)$.

Prava $y=1$ i parabola sijeku se u tački sa apscisom $\frac{1}{2}$; prava $y=1$ i luk $x=\sqrt{3-y^2}$ sijeku se u tački sa apscisom $\sqrt{2}$. Otuda je



Sl. 15

$$\int_0^{1/2} dx \int_0^{\sqrt{2x}} f(x, y)\, dy + \int_{1/2}^{1} dx \int_0^1 f(x, y)\, dy +$$

$$+ \int_{\sqrt{2}}^{\sqrt{3}} dx \int_0^{\sqrt{3-x^2}} f(x, y)\, dy.$$

#

Izračunati integral I:

97. $\displaystyle\iint\limits_{\substack{0\leqslant x\leqslant\frac{\pi}{2}\\ 0\leqslant y\leqslant\frac{\pi}{2}}} \sin(x+y)\,dxdy.$

98. $\displaystyle\iint\limits_{\substack{-1\leqslant x\leqslant 1\\ -2\leqslant y\leqslant 2}} \left(1-\frac{1}{3}x-\frac{1}{4}y\right)dxdy.$

99. $\displaystyle\iint\limits_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant y\leqslant x}} (x^2+y^2)\,dxdy.$

Rješenja:

97. Integral ćemo izračunati svođenjem na jednostruke integrale. Biće

$$I=\int_0^{\pi/2}dx\int_0^{\pi/2}\sin(x+y)\,dy=\int_0^{\pi/2}dx\,[-\cos(x+y)]\Big|_0^{\pi/2}=\int_0^{\pi/2}\left[\cos x-\cos\left(x+\frac{\pi}{2}\right)\right]dx=$$

$$=\sin x\Big|_0^{\pi/2}-\sin\left(x+\frac{\pi}{2}\right)\Big|_0^{\pi/2}=1+1=2.$$

98. $$I=\int_{-1}^{1}dx\int_{-2}^{2}\left(1-\frac{1}{3}x-\frac{1}{4}y\right)dy=\int_{-1}^{1}\left(y-\frac{1}{3}xy-\frac{1}{8}y^2\right)\Big|_{-2}^{2}dx=$$

$$=\int_{-1}^{1}\left(4-\frac{4}{3}x\right)dx=\left(4x-\frac{2}{3}x^2\right)\Big|_{-1}^{1}=8.$$

Ako se izvrši integracija najprije po x pa po y, dobiće se isti rezultat:

$$I=\int_{-2}^{2}dy\int_{-1}^{1}\left(1-\frac{1}{3}x-\frac{1}{3}y\right)dx=\int_{-2}^{2}\left(x-\frac{1}{6}x^2-\frac{1}{4}xy\right)\Big|_{-1}^{1}dy=$$

$$=\int_{-2}^{2}\left(2-\frac{1}{2}y\right)dy=\left(2y-\frac{1}{4}y^2\right)\Big|_{-2}^{2}=8.$$

99. $$I=\int_0^1dx\int_0^x(x^2+y^2)\,dy=\int_0^1dx\left(x^2y+\frac{y^3}{3}\right)\Big|_0^x=\int_0^1\left(x^3+\frac{x^3}{3}\right)dx=\frac{1}{3}.$$

#

Izračunati integral *I*:

$$\textbf{100.}\quad \iint\limits_{\substack{-3\leq x\leq 1\\ 2x-1\leq y\leq 2-x^2}} (x-y)\,dxdy\,.$$

$$\textbf{101.}\quad \iint\limits_{\substack{0\leq x\leq 1\\ 0\leq y\leq 1-x}} (2\,x^2+y^2+1)\,dxdy. \qquad \textbf{102.}\quad \iint\limits_{|x|+|y|\leq 1} x^2\,dxdy.$$

103. $\iint\limits_{D} (x+y)\,dxdy$, gdje je D oblast ograničena linijama $y=x^2$, $y=x$.

Rješenja:

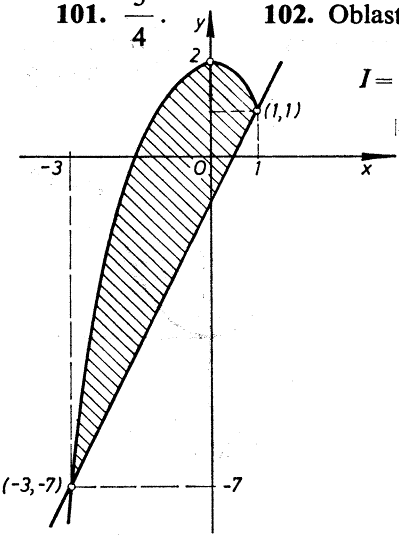
100. Prava $y=2\,x-1$ i parabola $y=2-x^2$ sijeku se u tačkama $(-3,\ -7)$, $(1,\ 1)$ (sl. 16). Biće:

$$I=\int\limits_{-3}^{1} dx\int\limits_{2x-1}^{2-x^2}(x-y)\,dy=\int\limits_{-3}^{1}\left(xy-\frac{y^2}{2}\right)\Bigg|_{2x-1}^{2-x^2}dx=$$

$$=\int\limits_{-3}^{1}\left(2\,x-x^3-2+2\,x^2-\frac{1}{2}\,x^4-2\,x^2+x+2x^2-2\,x+\frac{1}{2}\right)dx=$$

$$=\int\limits_{-3}^{1}\left(-\frac{1}{2}\,x^4-x^3+2\,x^2+x-\frac{3}{2}\right)dx=4\,\frac{4}{15}.$$

101. $\dfrac{3}{4}$.

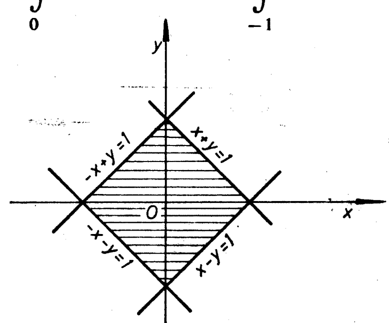


Sl. 16

102. Oblast integracije prikazana je na sl. 17. Biće:

$$I=\iint\limits_{|x|+|y|\leq 1} x^2\,dxdy=\int\limits_{0}^{1}x^2\,dx\int\limits_{x-1}^{1-x}dy+\int\limits_{-1}^{0}x^2\,dx\int\limits_{-1-x}^{1+x}dy=$$

$$=2\int\limits_{0}^{1}x^2\,(1-x)\,dx+2\int\limits_{-1}^{0}x^2\,(1+x)\,dx=\frac{1}{3}.$$



Sl. 17

$$\textbf{103.}\quad I=\int\limits_{0}^{1}dx\int\limits_{x^2}^{x}(x+y)\,dy=\int\limits_{0}^{1}\left(x^2+\frac{x^2}{2}-x^3-\frac{x^4}{2}\right)dx=\frac{3}{20}.$$

Zadaci za vježbu

U zadacima 3466 — 3476 proceniti date integrale.

3466. $\iint\limits_D (x+y+10)\,d\sigma$, gde je D—krug $x^2+y^2<4$.

3467. $\iint\limits_D (x^2+4y^2+9)\,d\sigma$, gde je D—krug $x^2+y^2<4$.

3468. $\iint\limits_D (x+y+1)\,d\sigma$, gde je D—pravougaonik $0<x<1$, $0<y<2$.

3469. $\iint\limits_D (x+xy-x^2-y^2)\,d\sigma$, gde je D—pravougaonik $0<x<1$, $0<y<2$.

3470. $\iint\limits_D xy(x+y)\,d\sigma$, gde je D—kvadrat $0\leqslant x\leqslant 2$, $0<y<2$.

3471. $\iint\limits_D (x+1)^y\,d\sigma$, gde je D—kvadrat $0\leqslant x\leqslant 2$, $0<y<2$.

3472. $\iint\limits_D (x^2+y^2-2\sqrt{x^2+y^2}+2)\,d\sigma$, gde je D—kvadrat $0<x<2$, $0<y<2$.

3473. $\iint\limits_D (x^2+y^2-4x-4y+10)\,d\sigma$, gde je D—oblast ograničena elipsom $x^2+4y^2-2x-16y+13=0$ (uključujući granicu).

U zadacima 3477 — 3484 izračunati date dvojne integrale po pravougaonim oblastima D koje su određene nejednakostima navedenim u zadacima.

3477. $\iint\limits_D xy\,dx\,dy \quad (0<x<1,\ 0<y<2)$.

3478. $\iint\limits_D e^{x+y}\,dx\,dy \quad (0<x<1,\ 0<y<1)$.

3479. $\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy \quad (0<x<1,\ 0<y<1)$.

3480. $\iint\limits_D \frac{dx\,dy}{(x+y+1)^2} \quad (0<x<1,\ 0\leqslant y<1)$.

3481. $\iint\limits_D \frac{y\,dx\,dy}{(1+x^2+y^2)^{\frac{3}{2}}} \quad (0<x<1,\ 0<y<1)$.

3482. $\iint\limits_D x\sin(x+y)\,dx\,dy \quad \left(0<x<\pi,\ 0<y<\frac{\pi}{2}\right)$.

3483. $\iint\limits_D x^2ye^{xy}\,dx\,dy \quad (0<x<1,\ 0<y<2)$.

3484. $\iint\limits_D x^2y\cos(xy^2)\,dx\,dy \quad \left(0<x<\frac{\pi}{2},\ 0<y<2\right)$

U zadacima 3485 — 3497 naći granice dvostrukog integrala na koji se svodi dvojni integral $\iint\limits_D f(x,y)\,dx\,dy$ za date (konačne) oblasti integracije D.

3485. Paralelogram koji obrazuju prave

$$x=3,\quad x=5,\quad 3x-2y+4=0,\quad 3x-2y+1=0.$$

3486. Trougao koji obrazuju prave $x=0$, $y=0$, $x+y=2$.

3487. $x^2+y^2\leqslant 1$, $x\geqslant 0$, $y\geqslant 0$.

3488. $x+y\leqslant 1$, $x-y\leqslant 1$, $x\geqslant 0$.

3489. $y\geqslant x^2$ $y<4-x^2$.

3490. $\frac{x^2}{4}+\frac{y^2}{9}\leqslant 1$. **3491.** $(x-2)^2+(y-3)^2\leqslant 4$.

Rješenja

3466. $8\pi(45-\sqrt{2})<I<8\pi(5+\sqrt{2})$.

3467. $36\pi<I<100\pi$.

3468. $2<I<8$. **3469.** $-8<I<\frac{2}{3}$.

2470. $0<I<64$.

3471. $4<I<36$.

3472. $4<I<8(5-2\sqrt{2})$. $4\pi<I<22\pi$.

3474. $0<I<\frac{4}{3}\pi R^5$.

3475. $24<I<72$.

3476. $29\pi\sqrt{3}<I<52\pi\sqrt{3}$.

3477. 1. **3478.** $(e-1)^2$.

3479. $\frac{\pi}{12}$.

3480. $\ln\frac{4}{3}$. **3481.** $\ln\frac{2+\sqrt{2}}{1+\sqrt{3}}$.

3482. $\pi-2$. **3483.** 2. **3484.** $-\frac{\pi}{16}$.

3485. $\int\limits_3^5 dx \int\limits_{\frac{3x+1}{2}}^{\frac{3x+4}{2}} f(x,y)\,dy$. **3486.** $\int\limits_0^2 dx \int\limits_0^{2-x} f(x,y)\,dy$.

3487. $\int\limits_0^1 dx \int\limits_0^{\sqrt{1-x^2}} f(x,y)\,dy$. **3488.** $\int\limits_0^1 dx \int\limits_{x-1}^{1-x} f(x,y)\,dy$.

3489. $\int\limits_{-\sqrt{2}}^{\sqrt{2}} dx \int\limits_{x^2}^{4-x^2} f(x,y)\,dy$. **3490.** $\int\limits_{-2}^{2} dx \int\limits_{-\frac{3}{2}\sqrt{4-x^2}}^{\frac{3}{2}\sqrt{4-x^2}} f(x,y)\,dy$.

3491. $\int\limits_0^4 dx \int\limits_{3-\sqrt{4x-x^2}}^{3+\sqrt{4x-x^2}} f(x,y)\,dy$.

3492. Oblast D je ograničena parabolama $y=x^2$ i $y=\sqrt{x}$.

3493. Trougao koji obrazuju prave $y=x$, $y=2x$ i $x+y=6$.

3494. Paralelogram koji obrazuju prave

$$y=x, \quad y=x+3, \quad y=-2x+1, \quad y=-2x+5.$$

3495. $y-2x<0$, $2y-x>0$, $xy<2$.

3496. $y^2<8x$, $y\leqslant 2x$, $y+4x-24<0$.

3497. Oblast D je ograničena hiperbolom $y^2-x^2=1$ i krugom $x^2+y^2=9$ (misli se na onu oblast u kojoj leži koordinatni početak).

U zadacima 3498 — 3503 promeniti redosled integracije u datim integralima.

3498. $\int_0^1 dy \int_y^{\sqrt{y}} f(x, y)\, dx$. **3499.** $\int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} f(x, y)\, dy$.

3500. $\int_0^r dx \int_x^{\sqrt{2rx-x^2}} f(x, y)\, dy$.

3501. $\int_{-2}^2 dx \int_{-\frac{1}{\sqrt{2}}\sqrt{4-x^2}}^{\frac{1}{\sqrt{2}}\sqrt{4-x^2}} f(x, y)\, dy$.

3502. $\int_1^2 dx \int_x^{2x} f(x, y)\, dy$. **3503.** $\int_0^2 dx \int_{2x}^{6-x} f(x, y)\, dy$.

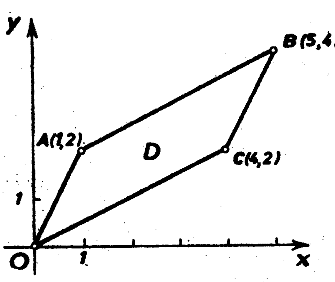
3504. Izmenivši redosled integracije predstaviti dati izraz u obliku jednog dvostrukog integrala:

1) $\int_0^1 dx \int_0^x f(x, y)\, dy + \int_1^2 dx \int_0^{2-x} f(x, y)\, dy$;

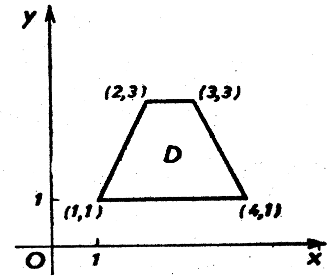
2) $\int_0^1 dx \int_0^{x^2} f(x, y)\, dy + \int_1^3 dx \int_0^{\frac{1}{2}(3-x)} f(x, y)\, dy$;

3) $\int_0^1 dx \int_0^{x^{\frac{2}{3}}} f(x, y)\, dy + \int_1^2 dx \int_0^{1-\sqrt{4x-x^2-3}} f(x, y)\, dy$.

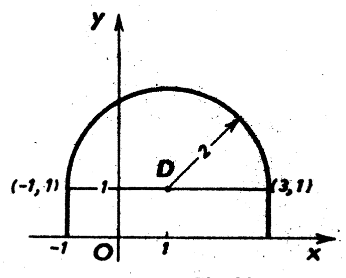
3505. Predstaviti dvojni integral $\iint\limits_D f(x, y)\, dx\, dy$ po oblastima D prikazanim na sl. 62, 63, 64, 65, — u obliku zbira dvostrukih integrala (sa najmanjim brojem sabiraka). Granične linije oblasti prikazanih na sl. 64 i 65 sastoje se iz pravolinijskih odsečaka i kružnih lukova.



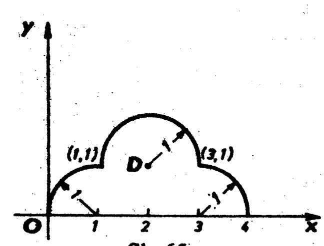
Sl. 62



Sl. 63



Sl. 64



Sl. 65

Rješenja

3492. $\int_0^1 dx \int_{x^2}^{\sqrt{x}} f(x, y)\, dy$.

3493. $\int_0^2 dx \int_x^{2x} f(x, y)\, dy + \int_2^3 dx \int_x^{6-x} f(x, y)\, dy$.

3494. $\int_{-\frac{2}{3}}^{\frac{1}{3}} dx \int_{1-2x}^{x+3} f(x, y)\, dy + \int_{\frac{1}{3}}^{\frac{2}{3}} dx \int_x^{x+3} f(x, y)\, dy + \int_{\frac{2}{3}}^{\frac{5}{3}} dx \int_x^{5-2x} f(x, y)\, dy$.

3495. $\int_0^1 dx \int_{\frac{x}{2}}^{2x} f(x, y)\, dy + \int_1^2 dx \int_{\frac{x}{2}}^{\frac{2}{x}} f(x, y)\, dy$.

3496. $\int_0^2 dx \int_{-2\sqrt{2x}}^{2x} f(x, y)\, dy + \int_2^{\frac{9}{2}} dx \int_{-2\sqrt{2x}}^{2\sqrt{2x}} f(x, y)\, dy + \int_{\frac{9}{2}}^8 dx \int_{-2\sqrt{2x}}^{24-4x} f(x, y)\, dy$.

3497. $\int_{-3}^{-2} dy \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\, dx + \int_{-2}^2 dx \int_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} f(x, y)\, dy + \int_2^3 dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\, dy$.

3498. $\int_0^1 dx \int_{x^2}^x f(x, y)\, dy$. **3499.** $\int_0^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)\, dx$.

3500. $\int_0^r dy \int_{r-\sqrt{r^2-y^2}}^y f(x, y)\, dx$. **3501.** $\int_{-\sqrt{2}}^{\sqrt{2}} dy \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} f(x, y)\, dx$.

3502. $\int_1^2 dy \int_1^y f(x, y)\, dx + \int_2^4 dy \int_{\frac{y}{2}}^2 f(x, y)\, dx$.

3503. $\int_0^4 dy \int_0^{\frac{y}{2}} f(x, y)\, dx + \int_4^6 dy \int_0^{6-y} f(x, y)\, dx$.

3504. 1) $\int_0^1 dy \int_y^{2-y} f(x, y)\, dx$; 2) $\int_0^1 dy \int_{\sqrt{y}}^{3-2y} f(x, y)\, dx$;

3) $\int_0^1 dy \int_{y^{\frac{3}{2}}}^{2-\sqrt{2y-y^2}} f(x, y)\, dx$.

3505. 1) $\int_0^2 dy \int_{\frac{y}{2}}^{2y} f(x, y)\, dx + \int_2^4 dy \int_{2y-3}^{\frac{y+6}{2}} f(x, y)\, dx$;

2) $\int_1^3 dy \int_{\frac{y+1}{2}}^{\frac{9-y}{2}} f(x, y)\, dx$; 3) $\int_{-1}^3 dx \int_0^{1+\sqrt{3+2x-x^2}} f(x, y)\, dy$;

4) $\int_0^1 dy \int_{1-\sqrt{1-y^2}}^{3+\sqrt{1-y^2}} f(x, y)\, dx + \int_1^2 dy \int_{2-\sqrt{2y-y^2}}^{2+\sqrt{2y-y^2}} f(x, y)\, dx$.

U zadacima 3506 — 3512 izračunati date integrale:

3506. 1) $\int\limits_0^a dx \int\limits_0^{\sqrt{x}} dy$; 2) $\int\limits_2^4 dx \int\limits_x^{2x} \frac{y}{x} dy$; 3) $\int\limits_1^2 dy \int\limits_0^{\ln y} e^x dx$.

3507. $\iint\limits_D x^3 y^2 \, dx \, dy$, D—krug $x^2+y^2<R^2$.

3508. $\iint\limits_D (x^2+y) \, dx \, dy$, oblast D je ograničena parabolama $y=x^2$ i $y^2=x$.

3509. $\iint\limits_D \frac{x^2}{y^2} \, dx \, dy$, oblast D je ograničena pravama $x=2$, $y=x$ i hiperbolom $xy=1$.

3510. $\iint\limits_D \cos(x+y) \, dx \, dy$, oblast D je ograničena pravama $x=0$, $y=\pi$ i $y=x$.

3511. $\iint\limits_D \sqrt{1-x^2-y^2}$ oblast D je četvrtina kruga $x^2+y^2<1$, koji leži u prvom kvadrantu.

3512. $\iint\limits_D x^2 y^2 \sqrt{1-x^3-y^3} \, dx \, dy$, oblast D je ograničena krivom $x^3+y^3=1$ i koordinatnim osama.

3513. Naći srednju vrednost funkcije $z=12-2x-3y$ u oblasti ograničenoj pravama $12-2x-3y=0$, $x=0$, $y=0$.

3514. Naći srednju vrednost funkcije $z=2x+y$ u oblasi ograničenoj pravom $x+y=3$ i koordinatnim osama.

3515. Naći srednju vrednost funkcije $z=x+6y$ u oblasti ograničenoj pravama $y=x$, $y=5x$ i $x=1$.

3516. Naći srednju vrednost funkcije $z=\sqrt{R^2-x^2-y^2}$ u krugu $x^2+y^2<R^2$.

Rješenja

3506. 1) $\frac{2}{3}a^{\frac{3}{2}}$; 2) 9; 3) $\frac{1}{2}$. **3507.** 0. **3508.** $\frac{33}{140}$. **3509.** $\frac{9}{4}$.

3510. -2. **3511,** $\frac{\pi}{6}$. **3512.** $\frac{4}{135}$. **3513.** 4. **3514.** 3. **3515.** $12\frac{2}{3}$.

3516. $\frac{2}{3}R$.

Smjena promjenjivih u dvostrukom integralu

Neka je dat integral $I=\iint\limits_D f(x,y)\,dx\,dy$.

Ako uvodimo nove promjenjive u i v takve da je
$x=\varphi(u,v)$
$y=\psi(u,v)$ tada se oblast D preslikava u D'. Jakobijan transformacijom $J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ i imamo

$$I=\iint\limits_{D'} f(\varphi(u,v),\ \psi(u,v))\,|J|\,du\,dv,$$

$$dx\,dy=|J|\,du\,dv$$

Npr. smjena polarnim koordinatama izgleda
$x=r\cos\varphi$
$y=r\sin\varphi$, r i φ su polarne koordinate, $r\geq 0$
$0\leq\varphi\leq 2\pi$

$$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix}=\begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}=r\cos^2\varphi+r\sin^2\varphi=r(\sin^2\varphi+\cos^2\varphi)=r$$

$$\iint\limits_D f(x,y)\,dx\,dy=\iint\limits_{D'} f(r\cos\varphi,\ r\sin\varphi)\cdot|r|\,d\varphi\,dr$$

Polarne koordinate obično uvodimo ako se u podintegralnoj f-ji ili u jednačinama koje opisuju oblast integracije pojavljuje izraz x^2+y^2.

Poopštene polarne koordinate izgledaju $x=ar\cos\varphi$ $(a>0)$
$y=br\sin\varphi$ $(b>0)$

$J=\ldots=abr$ (ostavljamo za vježbu kako doći do ovog rezultata)

Dati dvostruki integral $\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy$

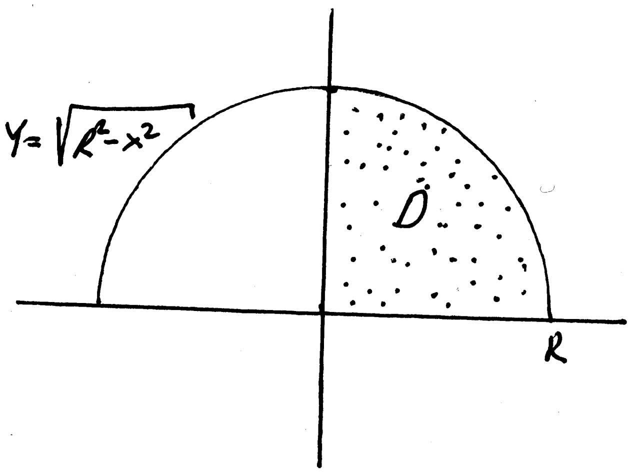
iz pravouglaonih koordinata transformisati na polarne koordinate.

Rj.

Oblast integracije D prema postavci zadatka je

$$D: \begin{cases} 0 \le x \le R \\ 0 \le y \le \sqrt{R^2-x^2} \end{cases}$$

Skicirajmo oblast D.



$$y^2 = R^2 - x^2$$

$$x^2 + y^2 = R^2$$

Polarne koordinate glase

$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$dx\,dy = r\,dr\,d\varphi$$

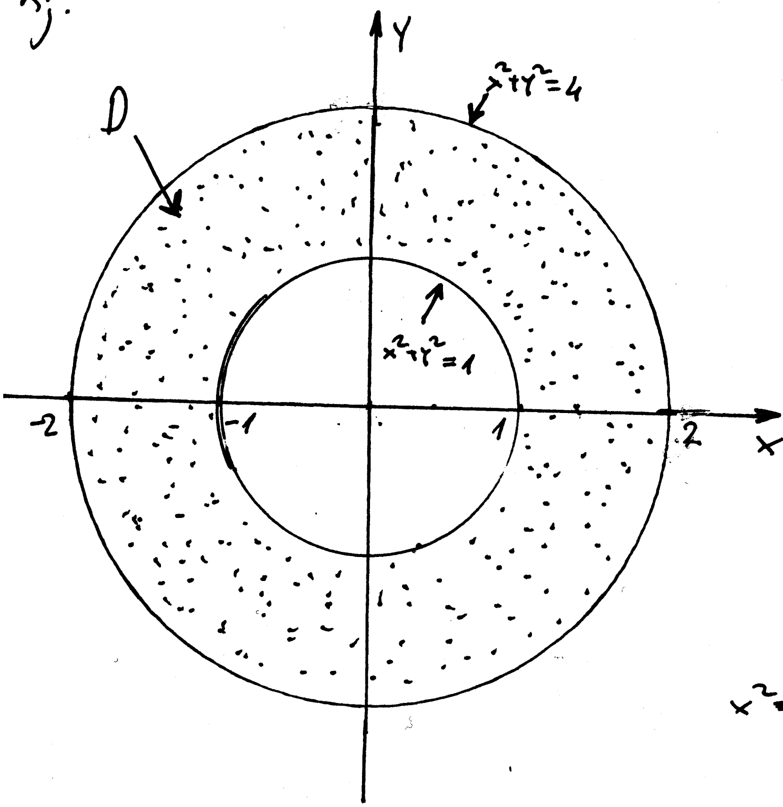
$$D \xrightarrow{\text{transformiše}} D': \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$

$$\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy = \int_0^R r\,dr \int_0^{\pi/2} f(r\cos\varphi, r\sin\varphi)\,d\varphi$$

(#) Izračunati dvostruki integral $I = \iint\limits_D \frac{dx\,dy}{\sqrt{x^2+y^2}}$ gdje je

D - kružni kolut, oblast ograđen kružnicama $x^2+y^2=1$ i $x^2+y^2=4$ (drugim riječima $D=\{(x,y) \mid x,y\in\mathbb{R} \text{ i } 1\le x^2+y^2\le 4\}$.

Rj.



Zadatak ćemo riješiti prelaskom na polarne koordinate

$x = r\cos\varphi$

$y = r\sin\varphi$

$dx\,dy = r\,dr\,d\varphi$

$D \xrightarrow{\text{transformiše}} D'$

gdje je

$$D' = \begin{cases} 1 \le r \le 2 \\ 0 \le \varphi \le 2\pi \end{cases}$$

$x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$

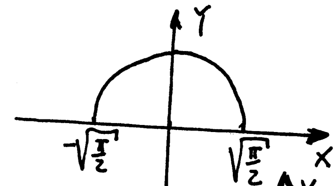
$$\iint\limits_D \frac{dx\,dy}{\sqrt{x^2+y^2}} = \iint\limits_{D'} \frac{r\,dr\,d\varphi}{\sqrt{r^2}} = \iint\limits_{D'} dr\,d\varphi = \int_1^2 dr \int_0^{2\pi} d\varphi = 2\pi \cdot 1 = 2\pi$$

Izračunati dvostruki integral

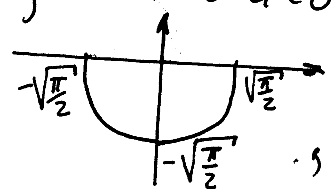
$$I=\int_0^{\sqrt{\frac{\pi}{2}}}dx\int_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}}\cos(x^2+y^2)\,dy\,.$$

Rj. Oblast integracije D je $D:\begin{cases}0\le x\le\sqrt{\frac{\pi}{2}}\\ -\sqrt{\frac{\pi}{2}-x^2}\le y\le\sqrt{\frac{\pi}{2}-x^2}\end{cases}$

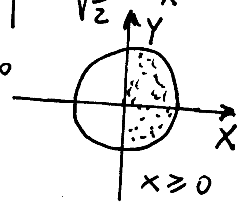
Znamo da f-ja $y=\sqrt{\frac{\pi}{2}-x^2}$ izgleda ovako



dok f-ja $y=-\sqrt{\frac{\pi}{2}-x^2}$ izgleda

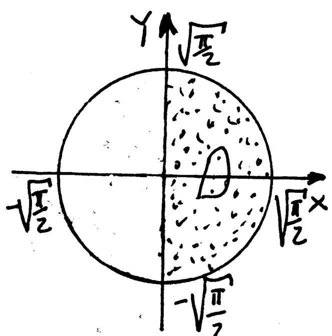


, sve zajedno $\Rightarrow$



.

Ove dvije f-je se dobiju iz f-je $y^2=\frac{\pi}{2}-x^2$ tj. $x^2+y^2=\frac{\pi}{2}$ što predstavlja jednačinu kruga sa centrom u koordinatnom početku, poluprečnika $\sqrt{\frac{\pi}{2}}$.



Uvedimo polarne koordinate

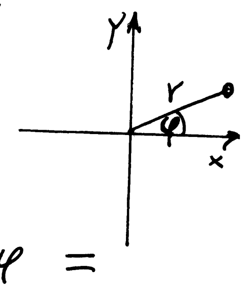
$x=r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

$D\xrightarrow{\text{transform.}}D':\begin{cases}0\le r\le\sqrt{\frac{\pi}{2}}\\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{cases}$

$x^2+y^2=\ldots=r^2$

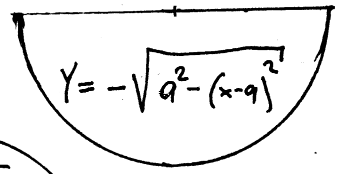


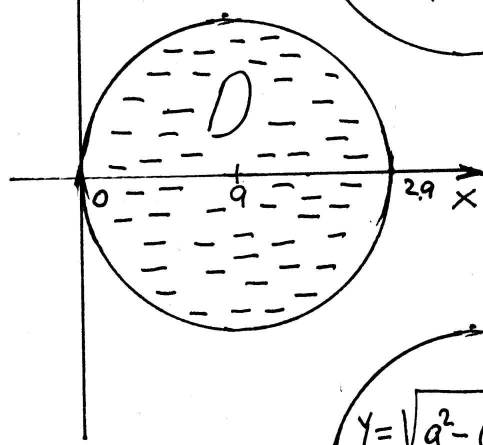
$$I=\int_0^{\sqrt{\frac{\pi}{2}}}dx\int_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}}\cos(x^2+y^2)\,dy=\iint_D\cos(x^2+y^2)\,dx\,dy=\iint_{D'}\cos(r^2)\,r\,dr\,d\varphi=$$

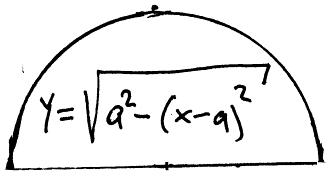
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^{\sqrt{\frac{\pi}{2}}}r\cos(r^2)\,dr=\left|\begin{array}{l}r^2=t\\ 2r\,dr=dt\\ r\,dr=\frac{1}{2}dt\\ r\Big|_0^{\sqrt{\frac{\pi}{2}}}\Rightarrow t\Big|_0^{\frac{\pi}{2}}\end{array}\right|=\frac{1}{2}\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{\pi/2}\cos t\,dt=\frac{1}{2}\cdot\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\sin t\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\cdot\pi\cdot 1=\frac{\pi}{2}$$ traženo rješenje

Izračunati $\iint\limits_D (x^2+y^2)\,dx\,dy$ gdje je D unutrašnjost kruga $x^2+y^2=2ax$.







Rj:

$$x^2+y^2=2ax$$

$$x^2-2ax+y^2=0$$

$$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$$

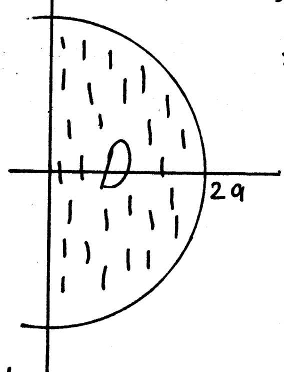
$$(x-a)^2+y^2=a^2$$

$S(a,0)$ centar

poluprečnik a

$$\iint\limits_D (x^2+y^2)\,dx\,dy=\int\limits_0^{2a}\left(\int\limits_{-\sqrt{a^2-(x-a)^2}}^{\sqrt{a^2-(x-a)^2}}(x^2+y^2)\,dy\right)dx=$$

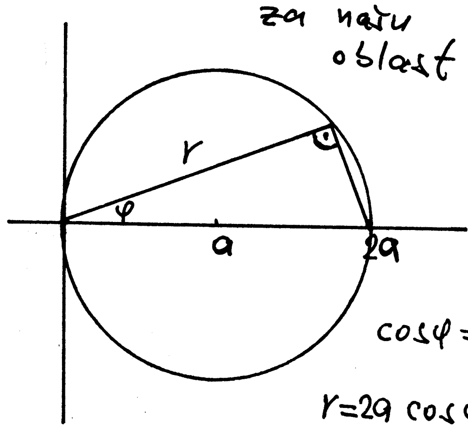
$$x=r\cos\varphi$$
$$y=r\sin\varphi$$
$$x^2+y^2=r^2$$



za ovakvu oblast imali bi

$$0\le r\le 2a$$
$$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$$

za našu oblast



$$\cos\varphi=\frac{r}{2a}$$

$$r=2a\cos\varphi$$

$$0\le r\le 2a\cos\varphi$$

$$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$$

$$dx\,dy=|J|\,dr\,d\varphi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r$$

$$=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int\limits_0^{2a\cos\varphi} r^2|r|\,dr\right]d\varphi=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int\limits_0^{2a\cos\varphi} r^3\,dr\right]d\varphi=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}r^4\Big|_0^{2a\cos\varphi}d\varphi=4a^4\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\varphi\,d\varphi$$

$$=\left|\cos^4\varphi=(\cos^2\varphi)^2=\left(\frac{1+\cos2\varphi}{2}\right)^2=\frac{1}{4}\left(\cos^2 2\varphi+2\cos2\varphi+1\right)\right|=$$

$$1=\sin^2\varphi+\cos^2\varphi$$
$$\cos2\varphi=\cos^2\varphi-\sin^2\varphi$$

$$= a^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 2\varphi + 2\cos 2\varphi + 1)\, d\varphi = a^4 \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 4\varphi)\, d\varphi + 2 \cdot \frac{1}{2} \sin 2\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right] \overset{(*)}{=}$$

$$1 = \sin^2 2\varphi + \cos^2 2\varphi$$

$$\cos 4\varphi = \cos^2 2\varphi - \sin^2 \varphi^2$$

$$1 + \cos 4\varphi = 2\cos^2 2\varphi$$

$$\cos^2 2\varphi = \frac{1}{2}(1+\cos 4\varphi)$$

$$\int \cos 2\varphi \, d\varphi = \left| \begin{array}{l} 2\varphi = t \\ 2d\varphi = dt \\ d\varphi = \frac{1}{2} dt \end{array} \right| = \frac{1}{2}\int \cos t \, dt$$

$$= \frac{1}{2}\sin t + c = \frac{1}{2}\sin 2\varphi + c$$

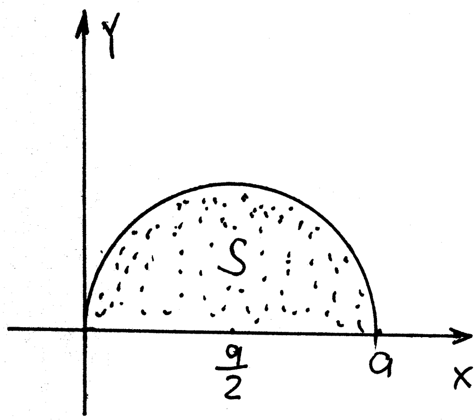
$$\overset{(*)}{=} a^4 \left[\frac{1}{2}\pi + \frac{1}{2}\cdot\frac{1}{4} \sin 4\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 0 + \pi \right] = a^4 \left[\frac{3\pi}{2} + \frac{1}{8}\cdot 0 \right] = \frac{3\pi}{2} a^4$$

II način: Uvodimo smjenu $x = a + r\cos\varphi$, $y = r\sin\varphi$, $0 \le \varphi \le 2\pi$, $0 \le r \le a$

URADITI ZA VJEŽBU

(#) Izračunati integral $\iint_S y\,dx\,dy$ gdje je S unutrašnjost gornjeg polukruga poluprečnika $\frac{a}{2}$ sa središtem u tački $\left(\frac{a}{2},0\right)$.

Rj.



I način:

$$\iint_S y\,dx\,dy = \dots = \int_0^{\frac{\pi}{2}}\left[\int_0^{a\cos\varphi} r\sin\varphi\cdot|r|\,dr\right]d\varphi$$

$$= \dots = \frac{a^3}{12}$$

OSTAVJAMO ZA VJEŽBU KAKO SMO OVO DOBILI

II način:

$$\iint_S y\,dx\,dy = \left|\begin{array}{l} x=\frac{a}{2}+r\cos\varphi \\ y=r\sin\varphi \\ 0\le\varphi\le\pi \\ 0\le r\le\frac{a}{2} \end{array}\right.$$

$dx\,dy=|J|\,dr\,d\varphi$

$$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi} \end{vmatrix}=\begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}$$

$$J=r$$

$$=\int_0^{\pi}\left[\int_0^{\frac{a}{2}} r\sin\varphi\cdot|r|\,dr\right]d\varphi=\int_0^{\pi}\sin\varphi\,\frac{1}{3}r^3\Big|_0^{\frac{a}{2}}\,d\varphi=\frac{a^3}{24}\int_0^{\pi}\sin\varphi\,d\varphi=$$

$$=\frac{a^3}{24}(-\cos\varphi)\Big|_0^{\pi}=-\frac{a^3}{24}(-1-1)=\frac{a^3}{12}$$

Izračunati dvostruki integral dat u polarnim koordinatama

$I = \iint\limits_D \rho \sin\varphi \, d\rho \, d\varphi$ gdje je oblast D

a) kružni sektor, ograničen linijama $\rho = a$, $\varphi = \frac{\pi}{2}$ i $\varphi = \pi$

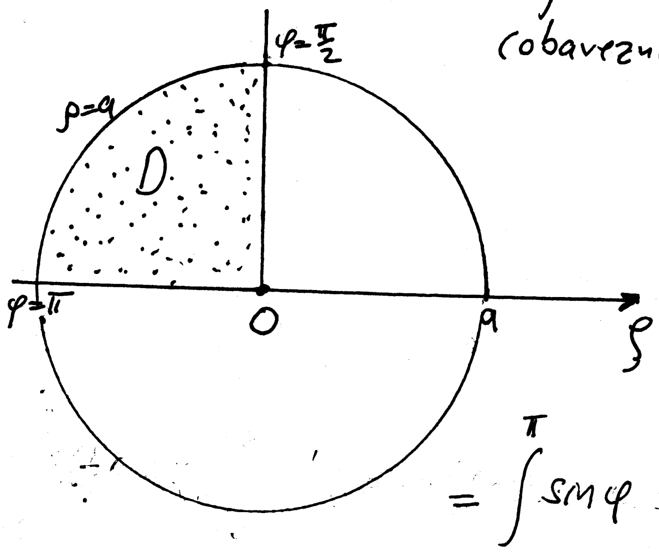
b) polukrug $\rho \le 2a\cos\varphi$, $0 \le \varphi \le \frac{\pi}{2}$

c) oblast između linija $\rho = 2 + \cos\varphi$ i $\rho = 1$.

(obavezno nacrtati izgled oblasti D)

Rj.

a)



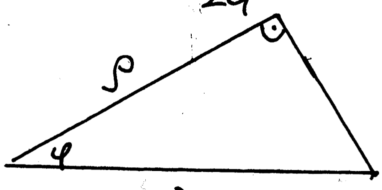
$$I = \iint\limits_D \rho \sin\varphi \, d\rho \, d\varphi = \int_{\frac{\pi}{2}}^{\pi} \sin\varphi \, d\varphi \int_0^a \rho \, d\rho =$$

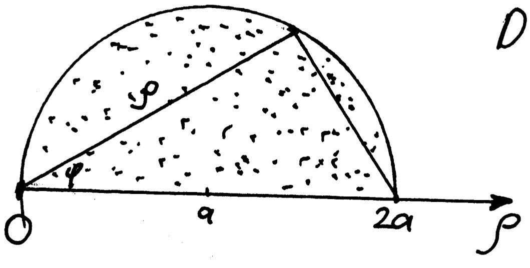
$$= \int_{\frac{\pi}{2}}^{\pi} \sin\varphi \left.\frac{\rho^2}{2}\right|_0^a d\varphi = \frac{a^2}{2}\left(-\cos\varphi\Big|_{\frac{\pi}{2}}^{\pi}\right) = \frac{a^2}{2}$$

b)

$\rho = 2a\cos\varphi$

$\cos\varphi = \frac{\rho}{2a}$





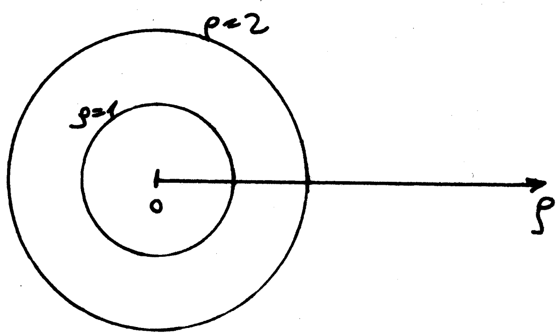
$$D: \begin{cases} 0 \le \rho \le 2a\cos\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$

$$I = \iint\limits_D \rho \sin\varphi \, d\rho \, d\varphi = \int_0^{\frac{\pi}{2}} \sin\varphi \, d\varphi \int_0^{2a\cos\varphi} \rho \, d\rho = \int_0^{\frac{\pi}{2}} \left.\frac{1}{2}\rho^2\right|_0^{2a\cos\varphi} \sin\varphi \, d\varphi =$$

$$= \frac{1}{2}\cdot 4a^2 \int_0^{\pi/2} \sin\varphi \cos^2\varphi \, d\varphi = \left| \begin{array}{l} \cos\varphi = t \\ -\sin\varphi \, d\varphi = dt \\ \varphi\Big|_0^{\frac{\pi}{2}} \Rightarrow t\Big|_1^0 \end{array} \right| = 2a^2\left(-\int_1^0 t^2 \, dt\right) =$$

$$= 2a^2 \int_0^1 t^2 \, dt = 2a^2 \cdot \left.\frac{t^3}{3}\right|_0^1 = \frac{2}{3}a^2$$ traženo rješenje

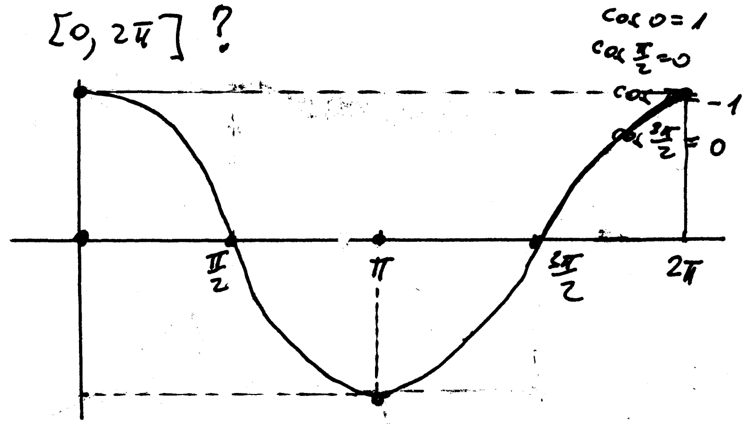
c) linije $\rho=1$ i $\rho=2$ nije teško nacrtati



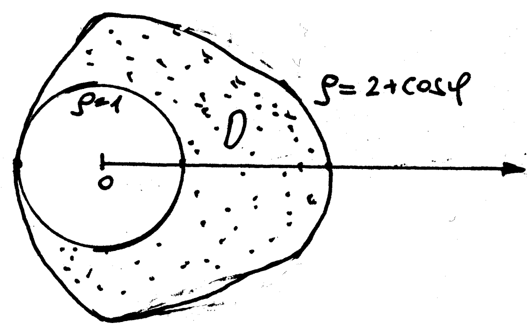
Problem predstavlja linija

$$\rho = 2+\cos\varphi$$

Kako izgleda $\cos\varphi$ na intervalu $[0, 2\pi]$?



Ako liniji $\rho=2$ dodamo $\cos\varphi$ imamo otprilike sljedeću sliku:



$$D: \begin{cases} 1 \leq \rho \leq 2+\cos\varphi \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$I=\iint_D \rho \sin\varphi \, d\rho \, d\varphi = \int_0^{2\pi} \sin\varphi \, d\varphi \int_1^{2+\cos\varphi} \rho \, d\rho = \int_0^{2\pi} \frac{\rho^2}{2}\Big|_1^{2+\cos\varphi} \sin\varphi \, d\varphi =$$

$$= \frac{1}{2}\int_0^{2\pi} \left((2+\cos\varphi)^2 - 1^2\right) \sin\varphi \, d\varphi = \frac{1}{2}\int_0^{2\pi} (4+4\cos\varphi+\cos^2\varphi-1) \sin\varphi \, d\varphi$$

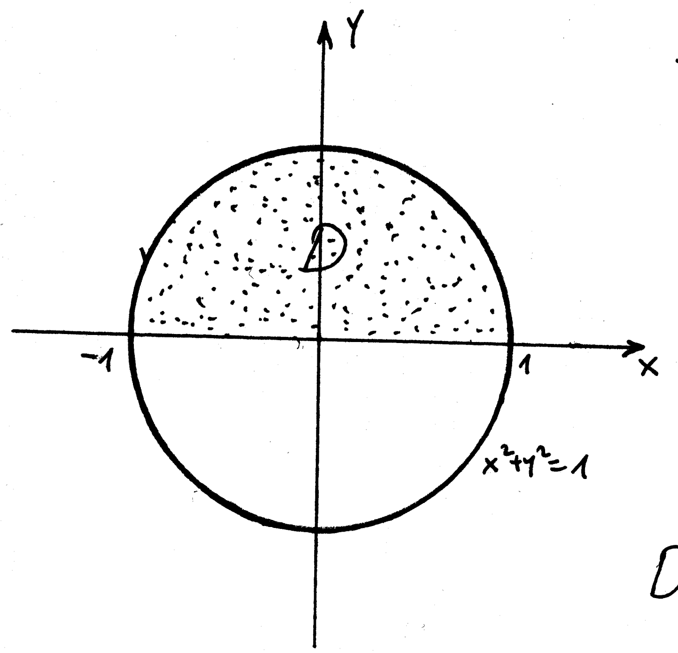
$$= -\frac{1}{2}\int_0^{2\pi} (\cos^2\varphi+4\cos\varphi+3) \, d\cos\varphi = \left(-\frac{1}{2}\right)\left(\frac{\cos^3\varphi}{3}\Big|_0^{2\pi} + 4\frac{\cos^2\varphi}{2}\Big|_0^{2\pi} + 3\cos\varphi\Big|_0^{2\pi}\right)$$

$$= \left(-\frac{1}{2}\right)\left(\frac{1}{3}(1-1) + 2(1-1) + 3(1-1)\right) = 0$$

traženo rješenje

Izračunati integral $I=\iint\limits_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$, ako je D oblast data sa: $x^2+y^2\le 1$, $y\ge 0$.

Rj. Skicirajmo oblast D



$$D:\begin{cases} -1\le x\le 1\\ 0\le y\le \sqrt{1-x^2}\end{cases}$$

Uvedimo polarne koordinate

$$x=r\cos\varphi$$
$$y=r\sin\varphi$$
$$dx\,dy=r\,dr\,d\varphi$$

$$D \xrightarrow{\text{transformace}} D', \quad D':\begin{cases} 0\le r\le 1\\ 0\le \varphi\le \pi\end{cases}$$

$$1-x^2-y^2=1-(x^2+y^2)=1-r^2$$
$$1+x^2+y^2=1+r^2$$

$$I=\iint\limits_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy=\iint\limits_{D'}\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr\,d\varphi=\int_0^{\pi}d\varphi\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr$$

Izračunajmo posebno drugi integral

$$\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr=\int_0^1\frac{\sqrt{1-r^2}}{\sqrt{1+r^2}}\,r\,dr=\int_0^1\frac{1-r^2}{\sqrt{(1+r^2)(1-r^2)}}\cdot r\,dr=\int_0^1\frac{r}{\sqrt{1-r^4}}\,dr-\int_0^1\frac{r^3}{\sqrt{1-r^4}}\,dr$$

$$\int_0^1\frac{r}{\sqrt{1-r^4}}\,dr=\left|\begin{matrix} r^2=t & r|_0^1\Rightarrow t|_0^1\\ 2r\,dr=dt & \\ r\,dr=\frac{1}{2}dt & \end{matrix}\right|=\frac{1}{2}\int_0^1\frac{dt}{\sqrt{1-t^2}}=\frac{1}{2}\arcsin t\Big|_0^1=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$

$$\int_0^1\frac{r^3}{\sqrt{1-r^4}}\,dr=\left|\begin{matrix} 1-r^4=s^2 & r^3dr=-\frac{1}{2}s\,ds\\ -4r^3dr=2s\,ds & r|_0^1\Rightarrow s|_1^0\end{matrix}\right|=-\frac{1}{2}\int_1^0\frac{s\,ds}{\sqrt{s^2}}=\frac{1}{2}$$

$$I=\int_0^{\pi}d\varphi\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr=\varphi\Big|_0^{\pi}\cdot\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{\pi^2}{4}-\frac{\pi}{2}$$ traženo rješenje

(#) Izračunati integral $I = \int_0^{\sqrt{3}/2} dx \int_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy$.

Rj: Pokušajmo prvo skicirati oblast integracije D. Primjetimo da se u drugom integralu pojavljuju f-je $y=\sqrt{1-x^2}$ i $y=1-\sqrt{1-x^2}$. Nacrtajmo ih.

$y=\sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2=1$

kruг sa centrom u $C(0,0)$ poluprečnika $r=1$

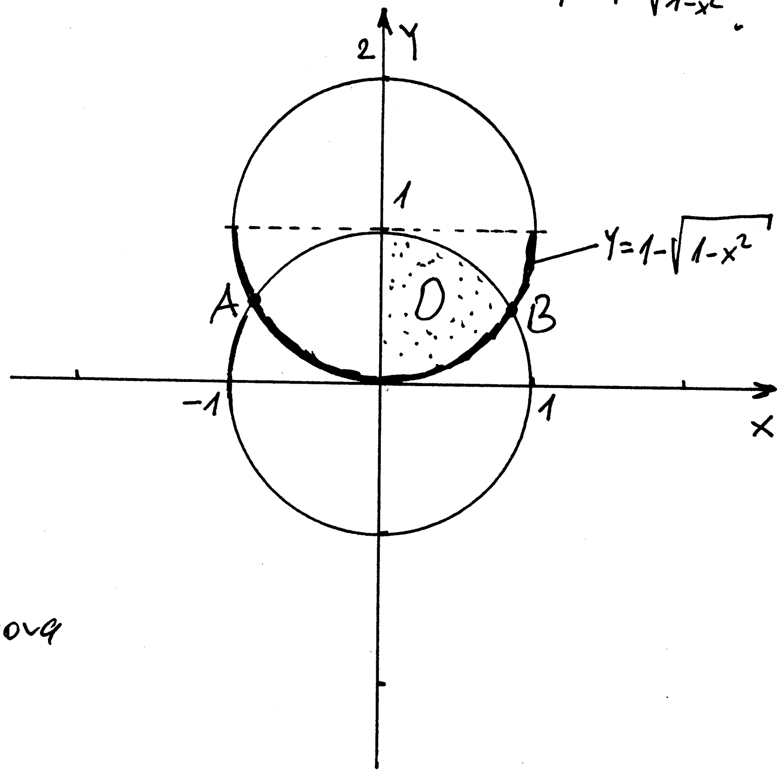
$y = 1-\sqrt{1-x^2}$

$y-1 = -\sqrt{1-x^2}$

$(y-1)^2 = 1-x^2$

$x^2+(y-1)^2=1$

krug sa centrom u $C(0,1)$ poluprečnika $r=1$



Pronađimo tačke presjeka ovih krugova

$$\begin{array}{l} x^2+y^2=1 \\ \underline{x^2+(y-1)^2=1} \\ x^2=1-y^2 \\ \underline{x^2+(y-1)^2=1} \end{array}$$

$1-y^2+(y-1)^2=1$

$1-\cancel{y^2}+\cancel{y^2}-2y+\cancel{1}=\cancel{1}$

$1-2y=0$

$2y=1$

$y=\frac{1}{2}$

$x^2+y^2=1$

$x^2=1-y^2$

$x^2=1-\frac{1}{4}$

$x^2=\frac{3}{4}$

$x_{1,2}=\pm\frac{\sqrt{3}}{2}$

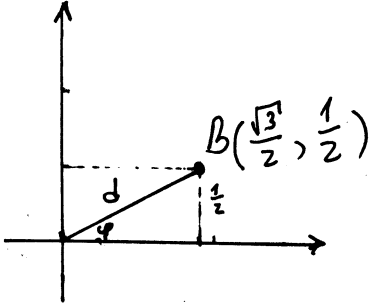
Tačke presjeka su $A(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ i $B(\frac{\sqrt{3}}{2}, \frac{1}{2})$

Sad možemo konačno nacrtati oblast integracije D.

$$I = \int_0^{\sqrt{3}/2} dx \int_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy = \iint_D \sqrt{x^2+y^2}\, dx\, dy$$

Oblast D ćemo podjeliti na dva dijela D_1 i D_2 pa ćemo imati

$$\iint_D \sqrt{x^2+y^2}\, dx\, dy = \iint_{D_1} \sqrt{x^2+y^2}\, dx\, dy + \iint_{D_2} \sqrt{x^2+y^2}\, dx\, dy.$$

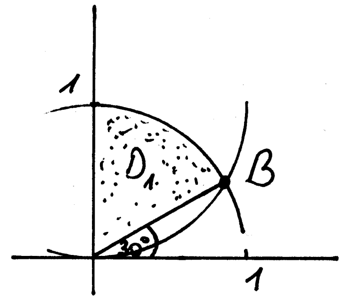


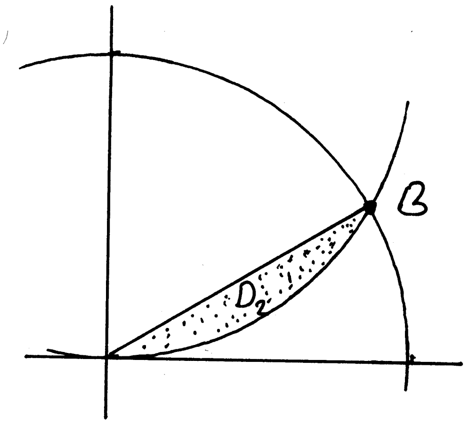
$d=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$

$\sin\varphi=\frac{\frac{1}{2}}{1}$

$\sin\varphi=\frac{1}{2}$

$\varphi=30^\circ$



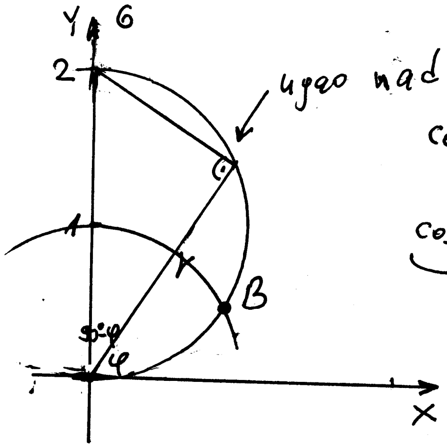


$\iint_{D_1}\sqrt{x^2+y^2}\,dx\,dy=$ | uvedimo polarne koordinate
$x=r\cos\varphi$
$y=r\sin\varphi$
$dx\,dy=r\,dr\,d\varphi$

$D_1 \xrightarrow{\text{transformiše}} D_1':\begin{cases}0\le r\le 1\\ \frac{\pi}{6}\le\varphi\le\frac{\pi}{2}\end{cases}$ |

$=\iint_{D_1'}\sqrt{r^2}\,r\,dr\,d\varphi=\int_0^1 r^2\,dr\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}d\varphi=$

$=\varphi\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cdot\frac{1}{3}r^3\Big|_0^1=\frac{1}{3}\left(\frac{3\pi}{6}-\frac{\pi}{6}\right)=\frac{1}{3}\cdot\frac{2\pi}{6}=\frac{1}{3}\cdot\frac{\pi}{3}=\frac{\pi}{9}$



$\cos(90^\circ-\varphi)=\frac{r}{2}$

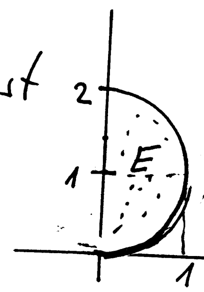
$\underbrace{\cos 90^\circ\cos\varphi+\sin 90^\circ\sin\varphi}_{\cos(90^\circ-\varphi)}=\frac{r}{2}$

$\sin\varphi=\frac{r}{2}$

$r=2\sin\varphi$

Prema tome pomoćna oblast E ima granice

$E:\begin{cases}0\le r\le 2\sin\varphi\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}$



Odavde možemo vidjeti polarne granice za D_2

$\iint_{D_2}\sqrt{x^2+y^2}\,dx\,dy=$ | uvedimo polarne koordinate
$x=r\cos\varphi$
$y=r\sin\varphi$
$dx\,dy=r\,dr\,d\varphi$

$D_2 \xrightarrow{\text{transformiše}} D_2':\begin{cases}0\le r\le 2\sin\varphi\\ 0\le\varphi\le\frac{\pi}{6}\end{cases}$ |

$=\iint_{D_2'}r^2\,dr\,d\varphi=\int_0^{\frac{\pi}{6}}d\varphi\int_0^{2\sin\varphi}r^2\,dr=\int_0^{\frac{\pi}{6}}\frac{1}{3}r^3\Big|_0^{2\sin\varphi}d\varphi=\frac{8}{3}\int_0^{\frac{\pi}{6}}\sin^3\varphi\,d\varphi=\overset{\text{za vježbu}}{\ldots}=-\sqrt{3}+\frac{16}{9}$

Prema tome $I=\frac{\pi}{9}+\frac{16}{9}-\sqrt{3}=\frac{\pi+16}{9}-\sqrt{3}$ traženo rješenje

Dati dvostruki integral $\displaystyle\int_{\frac{R}{2}}^{2R} dy \int_{0}^{\sqrt{2Ry-y^2}} f(x,y)\,dx$

iz pravouganih koordinata transformisati na polarne koordinate.

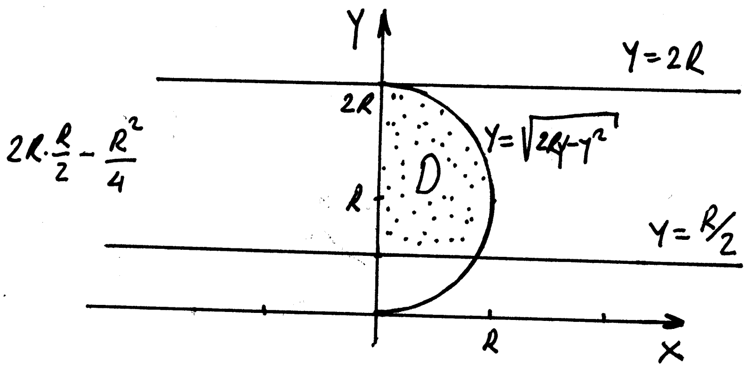
Rj: Skicirajmo oblast integracije.

Iz postavke vidimo da je x ograničen sa pravom $x=0$ i krivom $x=\sqrt{2Ry-y^2}$

$$D:\begin{cases} 0 \le x \le \sqrt{2Ry-y^2} \\ 2R \le y \le R/2 \end{cases}$$

$$x^2 = 2Ry - y^2$$
$$x^2 + y^2 - 2\cdot y\cdot R + R^2 - R^2 = 0$$
$$x^2 + (y-R)^2 = R^2$$

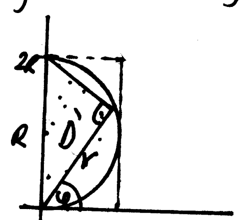
krug sa centrom u tački $(0,R)$ poluprečnika R.



Polarne koordinate glase

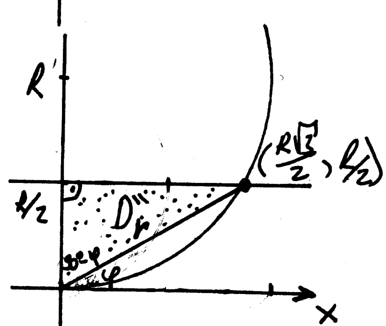
$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$dx\,dy = r\,dr\,d\varphi$$

Da bi došli do ideje kako opisati oblast D posmatrajmo sljedeće "jednostavnije" oblasti D' i D'':



$$\cos(90^\circ-\varphi) = \frac{r}{2R} \Rightarrow r = 2R\sin\varphi$$
$$\cos(90^\circ-\varphi) = \sin\varphi$$

$$D':\begin{cases} 0 \le r \le 2R\sin\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$



$$\cos(90^\circ-\varphi) = \frac{R/2}{r}$$
$$\sin\varphi = \frac{R}{2r}$$
$$2r = \frac{R}{\sin\varphi} \Rightarrow r = \frac{R}{2\sin\varphi}$$

$$D'':\begin{cases} 0 \le r \le \frac{R}{2\sin\varphi} \\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{2} \end{cases}$$

Sad nije teško vidjeti da će oblast D opisana pomoću polarnih koordinata postati

$$D: \begin{cases} \frac{R}{2\sin\varphi} \leq r \leq 2R\sin\varphi \\ \frac{\pi}{6} \leq \varphi \leq \frac{\pi}{2} \end{cases}$$

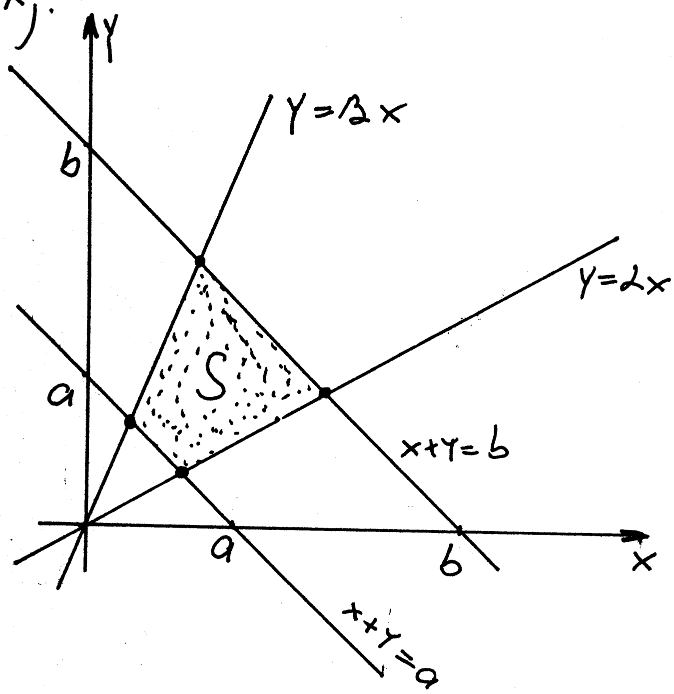
Prema tome

$$\int_{R/2}^{2R} dy \int_{0}^{\sqrt{2Ry-y^2}} f(x,y)\, dx = \int_{\pi/6}^{\pi/2} d\varphi \int_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi} f(r\cos\varphi, r\sin\varphi)\, r\, dr$$

Izračunati integral po oblasti S $\iint\limits_S \frac{1}{xy}\,dx\,dy$

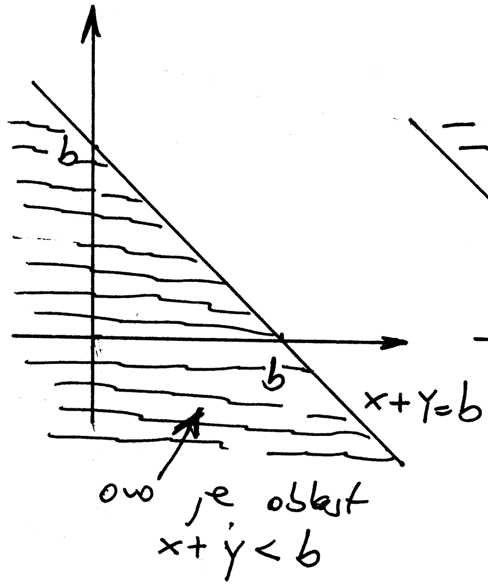
gdje je S oblast ograničena pravama $x+y=a$, $x+y=b$, $y=\alpha x$, $y=\beta x$ gdje su $0<a<b$ i $0<\alpha<\beta$.

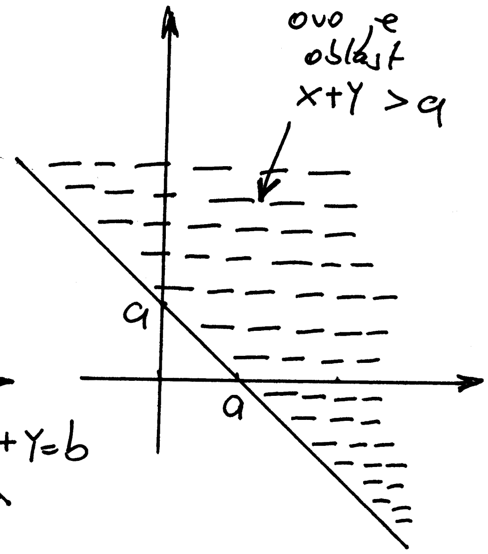
Rj.

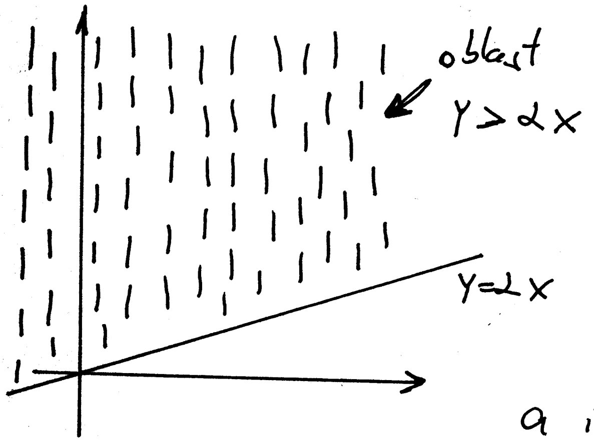


Na klasičan način ovaj zadatak nije lagano uraditi. Integral ćemo izračunati uvođenjem smjene.

Primjetimo







Iz ovoga možemo primjetiti da je S oblast gdje je $x+y$ između a i b a $\frac{y}{x}$ između α i β.

$$\iint\limits_S \frac{1}{xy}\,dx\,dy = \left| \begin{array}{ll} u=x+y & a\le u\le b \\ v=\frac{y}{x} & \alpha\le v\le\beta \end{array} \quad J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \begin{array}{l} y=vx \\ u=x+vx \\ u=x(1+v) \end{array} \right.$$

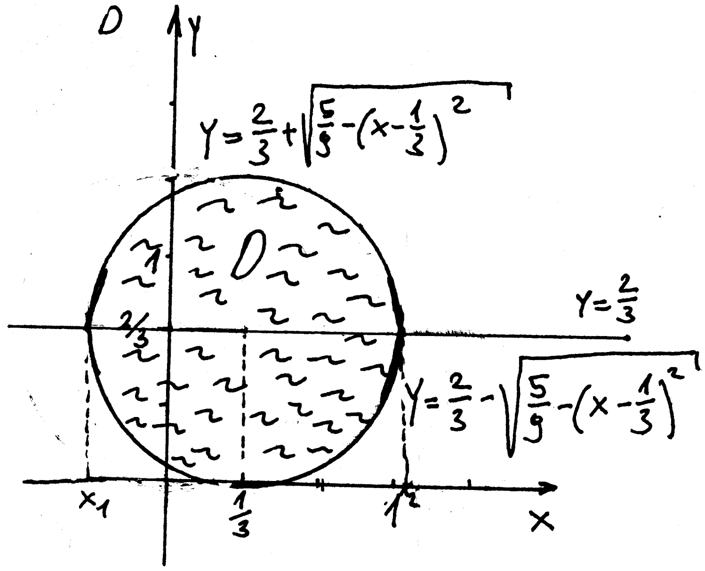
$$x=\frac{u}{1+v} \qquad y=\frac{uv}{1+v}$$

$$\frac{\partial x}{\partial u}=\frac{1}{1+v} \qquad \frac{\partial x}{\partial v}=u\cdot(-1)(1+v)^{-2}=\frac{-u}{(1+v)^2} \qquad dx\,dy=|J|\,du\,dv$$

$$\frac{\partial y}{\partial u}=\frac{v}{1+v} \qquad \frac{\partial y}{\partial v}=\frac{u(1+v)-uv\cdot 1}{(1+v)^2}=\frac{u}{(1+v)^2} \qquad J=\frac{u}{(1+v)^3}+\frac{uv}{(1+v)^3}=\frac{u}{(1+v)^2} \Bigg| =$$

$$=\int_a^b\left[\int_\alpha^\beta \frac{1}{\frac{u}{1+v}\cdot\frac{uv}{1+v}}\cdot\left|\frac{u}{(1+v)^2}\right|dv\right]du=\int_a^b\left[\int_\alpha^\beta \frac{1}{u}\cdot\frac{1}{v}\,dv\right]du=\int_a^b\frac{1}{u}\ln v\Big|_\alpha^\beta=\ln\frac{\beta}{\alpha}\cdot\ln u\Big|_a^b=\ln\frac{\beta}{\alpha}\ln\frac{b}{a}$$

(#) Izračunati dvostruki integral $I=\iint_D (x^2+y^2)\,dx\,dy$ gdje je $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le \frac{2}{3}(x+2y)\}$.



Rj. Odredimo šta je oblast D.

$$x^2+y^2 \le \frac{2}{3}(x+2y)$$
$$x^2+y^2 \le \frac{2}{3}x+\frac{4}{3}y$$
$$x^2-\frac{2}{3}x+y^2-\frac{4}{3}y \le 0$$
$$x^2-2\cdot x\cdot\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+y^2-2\cdot y\cdot\frac{2}{3}+\frac{4}{9}-\frac{4}{9} \le 0$$
$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{2}{3}\right)^2 \le \frac{5}{9}$$

D predstavlja unutrašnjost kruga s centrom u tački $(\frac{1}{3},\frac{2}{3})$ poluprečnika $r=\frac{\sqrt{5}}{3}\approx 0{,}74$

$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{2}{3}\right)^2=\frac{5}{9}$$
$$y=\frac{2}{3}$$
$$\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$$
$$x-\frac{1}{3}=\pm\frac{\sqrt{5}}{3}$$
$$x_{1,2}=\frac{1\pm\sqrt{5}}{3}$$

$$\boxed{\begin{aligned}\left(y-\frac{2}{3}\right)^2&=\frac{5}{9}-\left(x-\frac{1}{3}\right)^2\\ y&=\frac{2}{3}\pm\sqrt{\frac{5}{9}-\left(x-\frac{1}{3}\right)^2}\end{aligned}}$$

I način: klasičan način

Nađimo presječnu tačku kruga i prave $y=\frac{2}{3}$

$$I=\iint_D (x^2+y^2)\,dx\,dy=\int_{\frac{1-\sqrt{5}}{3}}^{\frac{1+\sqrt{5}}{3}}\left[\int_{\frac{2}{3}-\sqrt{\frac{5}{9}-(x-\frac{1}{3})^2}}^{\frac{2}{3}+\sqrt{\frac{5}{9}-(x-\frac{1}{3})^2}}(x^2+y^2)\,dy\right]dx=\dots$$

NA KLASIČAN NAČIN OVO JE TEŠKO URADITI

II način: Uvedimo neku smjenu promjenjivih. Kako je dat krug uvedimo polarne koordinate.

$$x=a+r\cos\varphi \qquad\qquad x=\frac{1}{3}+r\cos\varphi \qquad\qquad 0\le\varphi\le 2\pi$$
tj.
$$y=b+r\sin\varphi \qquad\qquad y=\frac{2}{3}+r\sin\varphi \qquad\qquad 0\le r\le\frac{\sqrt{5}}{3}$$

ove vrijednosti čitamo sa slike

$dx\,dy=r\,dr\,d\varphi$.

Jakobijan

$$dx\,dy=|J|\,dr\,d\varphi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r$$

$$x^2+y^2=\left(\frac{1}{3}+r\cos\varphi\right)^2+\left(\frac{2}{3}+r\sin\varphi\right)^2=\underline{\underline{\frac{1}{9}}}+\frac{2}{3}r\cos\varphi+\underline{r^2\cos^2\varphi}+\underline{\underline{\frac{4}{9}}}+\frac{4}{3}r\sin\varphi+\underline{r^2\sin^2\varphi}=$$
$$=\frac{5}{9}+r^2+\frac{2}{3}r\cos\varphi+\frac{4}{3}r\sin\varphi$$

$$I=\iint_D (x^2+y^2)\,dx\,dy=\iint_{D'}\left(\frac{5}{9}+r^2+\frac{2}{3}r(\cos\varphi+2\sin\varphi)\right)r\,dr\,d\varphi=\iint_{D'}\left(\frac{5}{9}r+r^3\right)dr\,d\varphi+$$
$$+\frac{2}{3}\iint_{D'}r^2(\cos\varphi+2\sin\varphi)\,dr\,d\varphi,$$

$$\iint_{D'}\left(\frac{5}{9}r+r^3\right)dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\frac{\sqrt{5}}{3}}\left(\frac{5}{9}r+r^3\right)dr\right]d\varphi=2\pi\cdot\left(\frac{5}{9}\cdot\frac{1}{2}r^2\Big|_0^{\frac{\sqrt{5}}{3}}+\right.$$

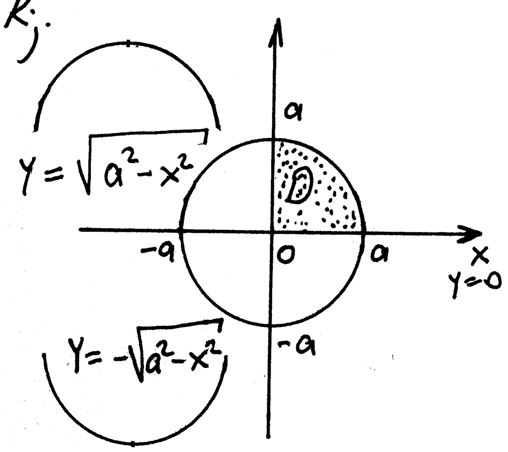
$$+\frac{1}{4}r^4\Big|_0^{\frac{\sqrt5}{3}}\Big) = 2\pi\left(\frac{5}{9\cdot 2}\cdot\frac{5}{9}+\frac{1}{4}\cdot\frac{5\cdot 5}{9\cdot 9}\right) = \pi\left(\frac{5^2}{9^2}+\frac{1}{2}\cdot\frac{5^2}{9^2}\right) = \frac{3}{2}\,\frac{5^2}{9^2}\pi = \frac{25}{54}\pi$$

$$\iint_{D'} r^2(\cos\varphi+2\sin\varphi)\,dr\,d\varphi = \int_0^{\sqrt5/3} r^2\left[\int_0^{2\pi}(\cos\varphi+2\sin\varphi)\,d\varphi\right] = \frac{r^3}{3}\Big|_0^{\sqrt5/3}\left(\sin\varphi\Big|_0^{2\pi} - 2\cos\varphi\Big|_0^{2\pi}\right) = 0$$

Prema tome $\displaystyle\iint_D (x^2+y^2)\,dx\,dy = \frac{25}{54}\pi$

(#) Izračunati $I = \displaystyle\iint_D \sqrt{x^2+y^2}\,dx\,dy$ gdje je D četvrtina kruga $x^2+y^2 \le a^2$.

Rj.



$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \int_0^a\left(\int_0^{\sqrt{a^2-x^2}}\sqrt{x^2+y^2}\,dy\right)dx =$$

$$= \left|\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ x^2+y^2 = r^2(\cos^2\varphi+\sin^2\varphi) = r^2 \end{array}\right. \qquad \begin{array}{c} 0\le x\le a \\ 0\le y\le \sqrt{a^2-x^2} \\ \Downarrow \\ 0\le\varphi\le\frac{\pi}{2} \\ 0\le r\le a \end{array}$$

$$dx\,dy = |J|\,dr\,d\varphi$$

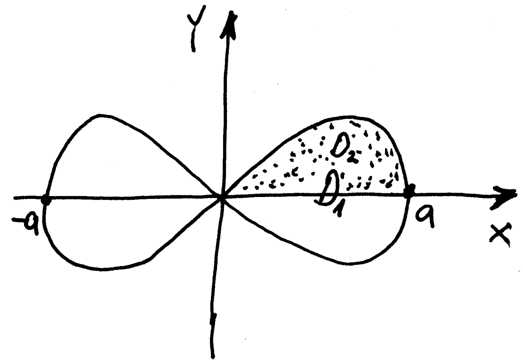
$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r \qquad = \int_0^a\left[\int_0^{\frac{\pi}{2}}\sqrt{r^2}\,|r|\,d\varphi\right]dr$$

$$= \int_0^a\left[\int_0^{\frac{\pi}{2}} r^2\,d\varphi\right]dr = \int_0^a r^2\cdot\varphi\Big|_0^{\frac{\pi}{2}}dr = \int_0^a\frac{\pi}{2}r^2\,dr = \frac{\pi}{2}\cdot\frac{1}{3}r^3\Big|_0^a = \frac{a^3\pi}{6}$$

(#) Izračunati dvostruki integral $\iint\limits_D dx\,dy$, ako je D oblast ograničena lemniskatom

$$(x^2+y^2)^2=a^2(x^2-y^2).$$

Rj. Lemniskata grafički izgleda ovako.



Pronađimo presječne tačke lemniskate sa x-osom: $y=0$

$$x^4=a^2x^2 \Rightarrow x^4-a^2x^2=0$$
$$x^2(x^2-a^2)=0$$
$$x_1=0,\ x_2=a,\ x_3=-a$$

Primjetimo da se površina oblasti D računa po formuli $P=\iint\limits_D dx\,dy$. Naša oblast D je simetrična u odnosu na y-osu pa je $\iint\limits_D dx\,dy = 2\iint\limits_{D_1} dx\,dy$.

Oblast D_1 je simetrična u odnosu na x-osu.

$$\iint\limits_D dx\,dy = 4\iint\limits_{D_2} dx\,dy$$

uvedimo polarne koordinate $x=r\cos\varphi$, $y=r\sin\varphi$, $dx\,dy=r\,dr\,d\varphi$

$$(x^2+y^2)^2=a^2(x^2-y^2)$$
$$(r^2)^2=a^2(r^2\cos^2\varphi-r^2\sin^2\varphi) \quad /:r^2\ (r\neq 0)$$
$$r^2=a^2(\cos^2\varphi-\sin^2\varphi)$$
$$r^2=a^2\cdot\cos 2\varphi \Rightarrow r=a\sqrt{\cos 2\varphi}$$

(primjetimo da za $\varphi>\frac{\pi}{4}$ r nije definisano)

$$D_2:\begin{cases}0<\varphi<\frac{\pi}{4}\\ 0<r<\sqrt{a^2\cos 2\varphi}\end{cases}$$

$$\iint\limits_D dx\,dy = 4\iint\limits_{D_2} r\,dr\,d\varphi = 4\int_0^{\frac{\pi}{2}} d\varphi\int_0^{a\sqrt{\cos 2\varphi}} r\,dr = 4\int_0^{\frac{\pi}{4}} \frac{1}{2}r^2\Big|_0^{a\sqrt{\cos 2\varphi}} d\varphi = \frac{4}{2}\int_0^{\frac{\pi}{4}} a^2\cos 2\varphi\,d\varphi =$$

$$= 2a^2\cdot\frac{1}{2}\sin 2\varphi\Big|_0^{\frac{\pi}{4}} = a^2\left(\sin\frac{\pi}{2}-0\right)=a^2$$ traženo rješenje

Oblast D preslikati pomoću preslikavanja f: $(x, y) \to (u\ v)$.

108. D je oblast ograničena parabolama $y^2 = px$, $y^2 = qx$, $x^2 = ay$, $x^2 = by$, $0 < p < q$, $0 < a < b$, a preslikavanje f je dato jednakostima $y^2 = ux$, $x^2 = vy$.

Rješenja:

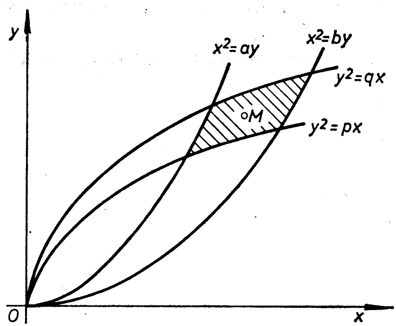
108. Odredićemo Jakobijan preslikavanja. Kako je $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$, to je:

$$\frac{D(u, v)}{D(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & -\frac{x^2}{y^2} \end{vmatrix} = 1 - \frac{4xy}{xy} = 1 - 4 = -3.$$

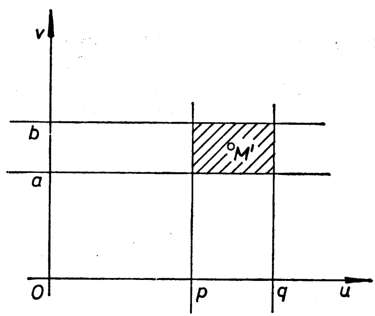
Dakle,

$$J = \frac{D(x, y)}{D(u, v)} = -\frac{1}{3} \neq 0,$$

pa je preslikavanje obostrano jednoznačno. Slike datih parabola su prave $u = p$, $u = q$, $v = a$, $v = b$, a oblast D (sl. 24) se preslikava na pravougaonik D' (sl. 25). Tačka $M \in D$ preslikava se na $M' \in D'$.



Sl. 24



Sl. 25

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Oblast D preslikati pomoću preslikavanja f: $(x, y) \to (u\ v)$.

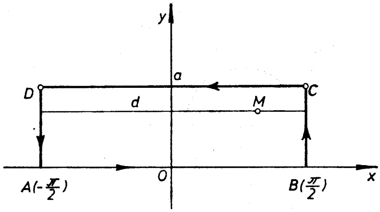
109. $D = \left\{(x, y): -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2},\ 0 \leqslant y \leqslant a\right\}$, a preslikavanje f je dato sa $u = \sin x \operatorname{ch} y$, $v = \cos x \operatorname{sh} y$.

Rješenja:

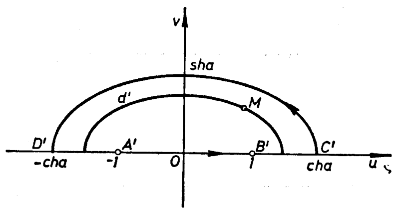
109. Preslikavanje je obostrano jednoznačno na $D \setminus \{A, B\}$, jer je

$$\frac{D(u, v)}{D(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos x \operatorname{ch} y & \sin x \operatorname{sh} y \\ -\sin x \operatorname{sh} y & \cos x \operatorname{ch} y \end{vmatrix} = \cos^2 x \operatorname{ch}^2 y + \sin^2 x \operatorname{sh}^2 y \neq 0$$

za $(x, y) \in D \setminus \{A, B\}$. Dio $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, $y = 0$ preslikava se na dio $-1 \leqslant u \leqslant 1$, $v = 0$ prave $v = 0$ (sl. 26 i 27). Duž BC ima jednačinu: $x = \frac{\pi}{2}$, $0 \leqslant y \leqslant a$, pa je njena slika skup tačaka (u, v) za koje je $u = \operatorname{ch} y$, $v = 0$.



Sl. 26



Sl. 27

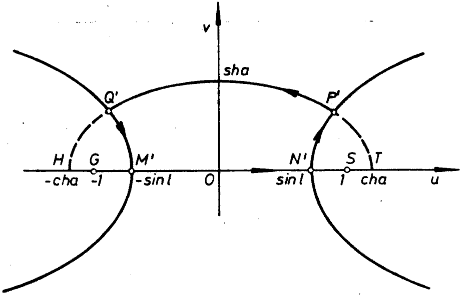
Duž DC ima jednačinu $y = a$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, pa se preslikava na skup tačaka (u, v) za koje je $u = \sin x \operatorname{ch} a$, $v = \cos x \operatorname{sh} a$, $v \geqslant 0$, tj. na gornju polovinu elipse

$$\frac{u^2}{\operatorname{ch}^2 a} + \frac{v^2}{\operatorname{sh}^2 a} = 1.$$

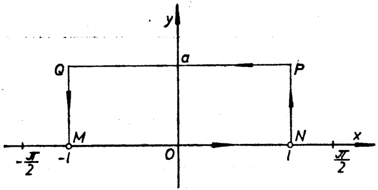
Duž DA preslikava se na duž $D'A'$ (sl. 26 i 27).

Da se unutrašnja tačka M oblasti D preslikava u unutrašnju tačku oblasti gornje poluelipse, može se zaključiti na sljedeći način. Kroz tačku M uočimo duž d paralelnu sa duži AB; njena slika će biti gornji luk elipse čije su poluose manje od $\operatorname{ch} a$ i $\operatorname{sh} a$, pa kako $M \in d \Rightarrow M' \in d'$, to slijedi zaključak.

Primjedba. Neka student sam nađe sliku pravougaonika $D = \{(x, y): -l \leqslant x \leqslant l,\ 0 \leqslant y \leqslant a,\ 0 \leqslant l \leqslant \frac{\pi}{2}$ (sl. 28 a i b). (Prava $x = l$ se preslikava na skup tačaka (u, v) za koje je $u = \sin l \operatorname{ch} y$, $v = \cos l \operatorname{sh} y$, tj. na skup tačaka (u, v) hiperbole $\frac{u^2}{\sin^2 l} - \frac{v^2}{\cos^2 l} = \operatorname{ch}^2 y - \operatorname{sh}^2 y = 1$.)



Sl. 28 a



Sl. 28 b

Kada $l \to \frac{\pi}{2}$, onda figura $M'N'P'Q'$ (sl. 28 a) postaje gornja poluelipsa, tj. $N'P'$ (luk hiperbole) teži duži ST.

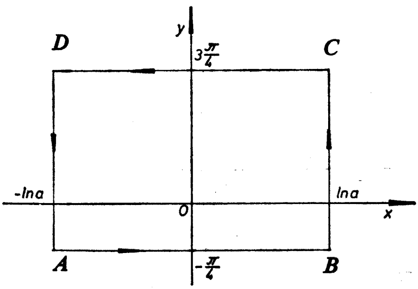
#

Oblast D preslikati pomoću preslikavanja f: $(x, y) \to (u\ v)$.

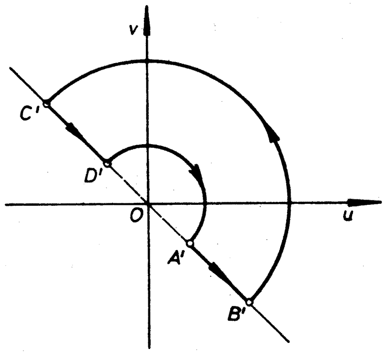
110. $D = \left\{(x, y) : |x| \leqslant \ln a,\ -\dfrac{1}{4}\pi \leqslant y \leqslant \dfrac{3}{4}\pi\right\}$, a preslikavanje f je dato sa $u = e^x \cos y$, $v = e^x \sin y$.

Rješenja:

110. Odredimo sliku konture oblasti $\mathcal{D}$ (sl. 29a).



Sl. 29a



Sl. 29b

Duž AB ima jednačinu $y = -\dfrac{\pi}{4}$, $-\ln a \leqslant x \leqslant \ln a$, pa će biti (sl. 29b).

$$A' \cdot B' = \left\{(u, v) : u = \frac{e^x}{\sqrt{2}},\ v = -\frac{e^x}{\sqrt{2}},\ -\ln a \leqslant x \leqslant \ln a\right\},$$

dakle, $A'B'$ je dio prave $v = -u$, pri čemu je $v < 0$, i to $-\dfrac{a}{\sqrt{2}} \leqslant v \leqslant -\dfrac{a^{-1}}{\sqrt{2}}$.

Na isti način zaključujemo da duž CD ima sliku $C'D'$, duž na pravoj $v = -u$,

$$\frac{a^{-1}}{\sqrt{2}} \leqslant v \leqslant \frac{a}{\sqrt{2}}.$$

Duž BC ima jednačinu $x = \ln a$, $-\dfrac{\pi}{4} \leqslant y \leqslant \dfrac{3\pi}{4}$, pa će njena slika biti skup tačaka $\{(u, v) : u = a \cdot \cos y,\ v = a\ \sin y\}$. Dakle, to je dio kružnice poluprečnika a.

Na isti način se zaključuje da duž DA ima kao sliku dio kružnice poluprečnika a^{-1}.

Preslikavanje je obostrano jednoznačno jer je

$$\frac{D(u, v)}{D(x, y)} = e^x > 0, \quad \text{za svako } x.$$

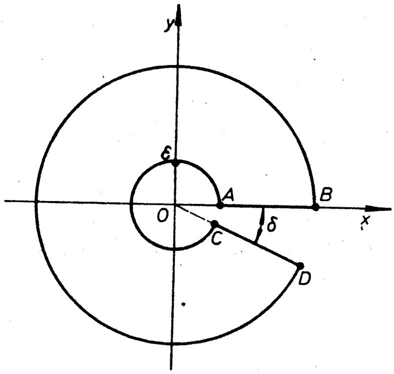
#

Oblast D preslikati pomoću preslikavanja f: $(x, y) \to (u\ v)$.

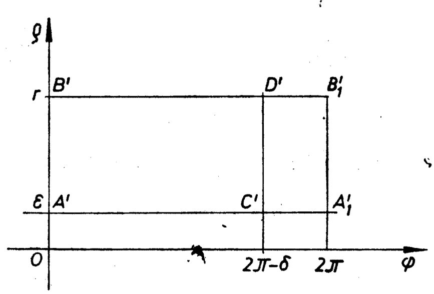
111. $D = \{(x, y) : x^2 + y^2 \leqslant r^2\}$, a preslikavanje f je dato sa $x = \rho \cos \varphi$, $y = \rho \sin \varphi$.

Rješenja:

111. Kako je $\dfrac{D(x, y)}{D(\rho, \varphi)} = \rho$, a u tački $(0, 0) \in D$ je $\rho = 0$, to ćemo najprije naći sliku oblasti $G \subset D$ koja je određena dijelovima kružnica poluprečnika r i ε, dužima AB i CD, pri čemu duž AB leži na x-osi, a duž CD na polupravoj čija je početna tačka $O(0, 0)$ i koja gradi ugao $2\pi - \delta$ (odnosno δ) sa polupravom OB (sl. 30a). Oblast G se preslikava na pravougaonik $A'B'D'C'$ (sl. 31a).



Sl. 30a

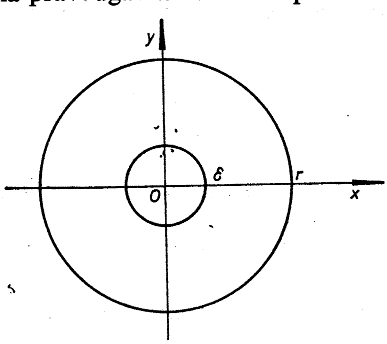


Sl. 31a

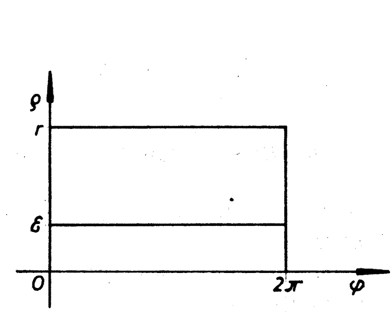
Ako pustimo da $\delta \to 0$, onda tačka $C \to A$, $D \to B$, i $D' \to B_1'$, $C' \to A_1'$.

Dakle, duži AB u ovom preslikavanju odgovaraju i duž $A'B'$ i duž $A_1' B_1'$.

Kružni prsten određen kružnicama poluprečnika r i ε preslikava se na pravougaonik određen pravama $\rho = \varepsilon$, $\rho = r$, $\varphi = 0$, $\varphi = 2\pi$ (sl. 30b, 31b).



Sl. 30b



Sl. 31b

Ako sada pustimo da $\varepsilon \to 0$, onda slika ε kružnice (duž) teži duži $[0, 2\pi]$ na pravoj $\rho = 0$ u sistemu $O\rho\varphi$. To znači da u ovom preslikavanju tački $(0, 0)$ odgovara duž $[0, 2\pi]$. Krug poluprečnika r preslikava se na pravougaonik $\rho = 0$, $\rho = r$, $\varphi = 0$, $\varphi = 2\pi$.

Primjedba. Neka student uoči značenja veličina ρ i φ u koordinatnom sistemu Oxy.

\# Pomoću smjene promjenljivih izračunati integrale:

114. $\iint\limits_D \sqrt{r^2-(x^2+y^2)}\,dxdy$, gdje je D oblast ograničena kružnicom $x^2+y^2-rx=0$.

115. $\iint\limits_D \ln(x^2+y^2)\,dxdy$, gdje je D oblast ograničena kružnicama $x^2+y^2=e^2$ i $x^2+y^2=e^4$.

Rješenja:

114. Uvodeći smjenu promjenljivih $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, podintegralna funkcija postaje $\sqrt{a^2-\rho^2}$, pa kako je $|J|=\rho$, biće

$$I=\iint\limits_{D'} \sqrt{a^2-\rho^2}\;\rho\,d\rho\,d\varphi.$$

Jednačina kružnice u novim koordinatama je:

$$x^2+y^2-rx=\rho^2\cos^2\varphi+\rho^2\sin^2\varphi-r\rho\cos\varphi=0,$$

tj. $\rho=r\cos\varphi$.

Otuda je

$$I=\int\limits_{-\pi/2}^{\pi/2} d\varphi\int\limits_0^{\rho(\varphi)} \sqrt{r^2-\rho^2}.\;\rho\,d\rho=\int\limits_{-\pi/2}^{\pi/2} d\varphi\int\limits_0^{r\cos\varphi} \rho\sqrt{r^2-\rho^2}\,d\rho=$$

$$=-\frac{1}{3}\int\limits_{-\pi/2}^{\pi/2}(r^2-\rho^2)^{3/2}\Big|_0^{r\cos\varphi}d\varphi=\frac{r^3}{3}\int\limits_{+\pi/2}^{\pi/2}(1-\sin^3\varphi)\,d\varphi=\frac{r^3\pi}{3}.$$

115. Smjenom $x=\rho\cos\varphi$, $y=\rho\sin\varphi$ dobija se

$$\iint\limits_D \ln(x^2+y^2)\,dxdy=2\iint\limits_{D'}\rho\ln\rho\;d\rho\;d\varphi=2\int\limits_0^{2\pi}d\varphi\int\limits_e^{e^2}\rho\ln\rho\;d\rho=$$

$$=4\pi\int\limits_e^{e^2}\rho\ln\rho\;d\rho=4\pi\left[\frac{1}{2}\rho^2\ln\rho-\frac{1}{4}\rho^2\right]_e^{e^2}=\pi e^2(3e^2-1).$$

(Za izračunavanje integrala $\int\rho\ln\rho\,d\rho$ primijenjena je parcijalna integracija.)

$\#$ Pomoću smjene promjenljivih izračunati integrale:

116. $I(r)=\iint\limits_D e^{-x^2-y^2}\,dxdy$, gdje je D oblast ograničena kružnicom $x^2+y^2=r^2$. Naći $\lim I(r)$ kad $r\to\infty$.

117. $\displaystyle\iint\limits_D \frac{dxdy}{(x^2+y^2)\left(1+\sqrt[3]{x^2+y^2}\right)}$. $D=\{(x,y): x^2-y^2\leqslant 0,\ 1\leqslant x^2+y^2\leqslant 4\}$.

Rješenja: **116.** $I(r)=\int\limits_0^{2\pi} d\varphi\int\limits_0^r e^{-\rho^2}\rho\,d\rho=(1-e^{-r^2})\pi$,

$\lim\limits_{r\to\infty} I(r)=\pi$. Ovo znači da je

$$\left(\int_{-\infty}^{\infty} e^{-t^2}\,dt\right)^2=\int_{-\infty}^{\infty} e^{-x^2}dx\int_{-\infty}^{\infty} e^{-y^2}dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy=\pi,$$

tj.

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt=\sqrt{\pi}.$$

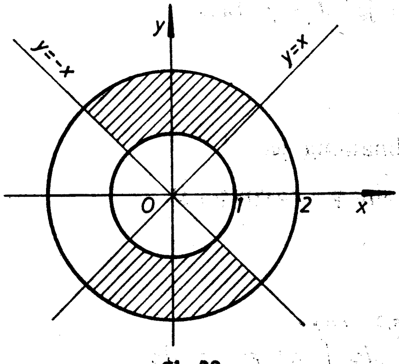
117. Najprije skiciraćemo oblast integracije. Biće:

$\{(x,y): x^2-y^2\leqslant 0\}=\{(x,y): (x-y)(x+y)\leqslant 0\}=\{(x,y): x<y\wedge x>-y$ ili $x>y\wedge x<-y\}$,

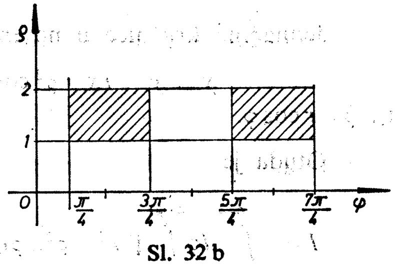
odnosno

$$\{(x,y): x^2-y^2\leqslant 0\}=\{(x,y): x^2\leqslant y^2\}=\{(x,y): |x|\leqslant|y|\}.$$

Oblast integracije D prikazana je na sl. 32a.



Sl. 32a



Sl. 32b

Smjenom $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, data oblast D preslikava se na oblast D' (sl. 32a i b), pa je:

$$I=\iint\limits_{D'}\frac{d\rho\,d\varphi}{\rho\left(1+\sqrt[3]{\rho^2}\right)}=\int_{\pi/4}^{3\pi/4} d\varphi\int_1^2\frac{d\rho}{\rho\left(1+\sqrt[3]{\rho^2}\right)}+\int_{5\pi/4}^{7\pi/4} d\varphi\int_1^2\frac{d\rho}{\rho\left(1+\sqrt[3]{\rho^2}\right)}=$$

$$=\pi\int_1^2\frac{d\rho}{\rho\left(1+\sqrt[3]{\rho^2}\right)}.$$

Smjenom $\sqrt[3]{\rho^2}=t$ dobija se

$$I=\frac{3\pi}{2}\int_1^{\sqrt[3]{4}}\frac{1}{t(t+1)}\,dt=\frac{3\pi}{2}\ln\frac{t}{t+1}\Bigg|_1^{\sqrt[3]{4}}=\frac{\pi}{2}\ln\frac{32}{\left(1+\sqrt[3]{4}\right)^3}.$$

\# Pomoću smjene promjenljivih izračunati integral:

120. $\iint\limits_D (x+y)^p (x-y)^q \, dxdy$, D je oblast ograničena pravama $x+y=1$, $x-y=1$, $x+y=3$, $x-y=-1$, p realan a q prirodan broj.

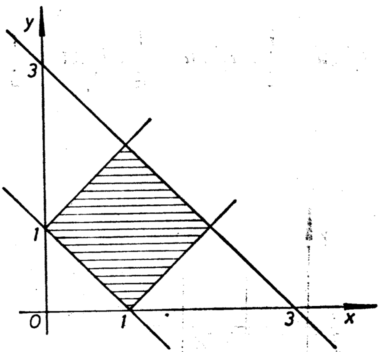
Rješenja:

120. Koristićemo smjenu

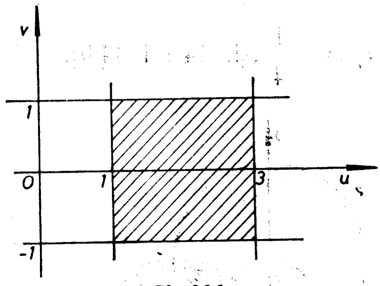
$$x+y=u, \quad x-y=v \Leftrightarrow x=\frac{1}{2}(u+v), \quad y=\frac{1}{2}(u-v).$$

Oblast D (kvadrat na sl. 33a) preslikava se na kvadrat D' (sl. 33b.); preslikavanje je obostrano jednoznačno, jer je

$$J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}=-\frac{1}{2}\,(\neq 0).$$



Sl. 33 a



Sl. 33 b

Biće: $I=\iint\limits_{D'} u^p v^q |J| \, dudv=\frac{1}{2}\int\limits_1^3 u^p \, du \int\limits_{-1}^1 v^q \, dv=\frac{1}{2}\,\frac{u^{p+1}}{p+1}\Bigg|_1^3 \cdot \frac{v^{q+1}}{q+1}\Bigg|_{-1}^1=$

$$=\frac{1}{2(p+1)(q+1)}\cdot(3^{p+1}-1)\cdot[1-(-1)^{q+1}] \text{ za } p\neq -1, \quad q\neq -1.$$

Konačno, $I=0$ za $q=2k-1$, $\pm k=1, 2, \ldots$; $I=\dfrac{3^{p+1}-1}{(p+1)(q+1)}$ za $q=2k$, $\pm k=0, 1, 2, \ldots$

Neka student samostalno riješi slučaj $p=-1 \vee q=-1$.

\# Pomoću smjene promjenljivih izračunati integral:

122. $\iint\limits_D (x^2+y^2)^{-2}\,dxdy$, gdje je D oblast ograničena kružnicama $l_1: x^2+y^2-2x=0$, $l_2: x^2+y^2-4x=0$; $l_3: x^2+y^2-2y=0$, $l_4: x^2+y^2-4y=0$.

Rješenja:

122. Napisaćemo jednačine kružnica u obliku

$$l_1: 1-2\frac{x}{x^2+y^2}=0, \quad l_2: 1-4\frac{x}{x^2+y^2}=0,$$

$$l_3: 1-2\frac{y}{x^2+y^2}=0, \quad l_4: 1-4\frac{y}{x^2+y^2}=0$$

i koristićemo smjenu

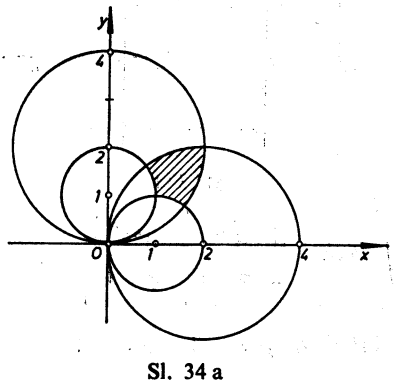
$$\frac{x}{x^2+y^2}=u, \ \frac{y}{x^2+y^2}=v \Leftrightarrow \frac{u}{u^2+v^2}=x, \ \frac{v}{u^2+v^2}=y.$$

Pri tome je

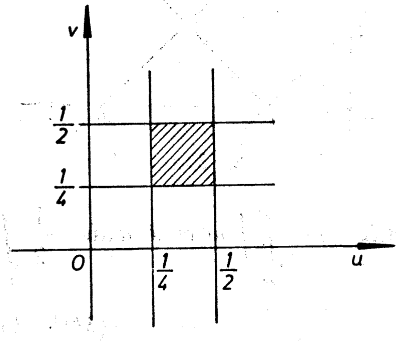
$$\frac{D(x,y)}{D(u,v)}=-\frac{1}{(u^2+v^2)^2}, \quad u^2+v^2=\frac{1}{x^2+y^2}.$$

Sada je $$I=\iint\limits_{D'} dudv,$$

pri čemu je oblast D' ograničena pravama $l_1': u=\frac{1}{2}$, $l_2': u=\frac{1}{4}$, $l_3': v=\frac{1}{2}$, $l_4': v=\frac{1}{4}$ (sl. 34 a i 34 b).



Sl. 34 a



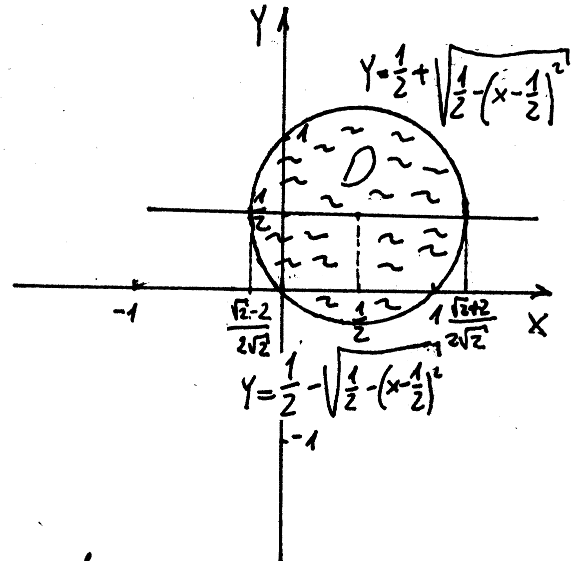
Sl. 34 b

Biće

$$I=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}.$$

\# Izračunati dvostruki integral: $I=\iint\limits_{D}(x+y)\,dx\,dy$, gdje je

$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le x+y\}$.



Rj.

$$x^2+y^2\le x+y$$

$$x^2-x+y^2-y\le 0$$

$$x^2-2\cdot x\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+y^2-2\cdot y\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}\le 0$$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\le\frac{1}{2}$$

Unutrašnjost kruga sa centrom u tački $S\left(\frac{1}{2},\frac{1}{2}\right)$

poluprečnika $r=\frac{1}{\sqrt{2}}\approx 0{,}7$.

Nađimo presječne tačke kruga sa pravom $y=\frac{1}{2}$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2},\quad y=\frac{1}{2}$$

$$\left(x-\frac{1}{2}\right)^2=\frac{1}{2} \qquad x-\frac{1}{2}=\pm\frac{1}{\sqrt{2}}$$

$$x_1=\frac{1}{2}+\frac{1}{\sqrt{2}}=\frac{2+\sqrt{2}}{2\sqrt{2}} \qquad x_2=\frac{1}{2}-\frac{1}{\sqrt{2}}=\frac{\sqrt{2}-2}{2\sqrt{2}}$$

I način:

$$I=\iint\limits_{D}(x+y)\,dx\,dy=\int_{\frac{\sqrt{2}-2}{2\sqrt{2}}}^{\frac{\sqrt{2}+2}{2\sqrt{2}}}\left[\int_{\frac{1}{2}-\sqrt{\frac{1}{2}-(x-\frac{1}{2})^2}}^{\frac{1}{2}+\sqrt{\frac{1}{2}-(x-\frac{1}{2})^2}}(x+y)\,dy\right]dx=\ldots$$

KOMPLIKOVANO

$$\left(y-\frac{1}{2}\right)^2=\frac{1}{2}-\left(x-\frac{1}{2}\right)^2$$

$$y-\frac{1}{2}=\pm\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}$$

$$\boxed{y=\frac{1}{2}\pm\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}}$$

II način: Uvedimo neku smjenu promjenljivih.

Kako je u pitanju krug, uvedimo polarne koordinate.

$$x=a+r\cos\varphi \qquad x=\frac{1}{2}+r\cos\varphi \qquad 0\le\varphi\le 2\pi$$

$$y=b+r\sin\varphi \quad \text{tj.} \quad y=\frac{1}{2}+r\sin\varphi \qquad 0\le r\le\frac{\sqrt{2}}{2}$$

ove vrijednosti čitamo sa slike

Jakobijan

$$dx\,dy=|J|\,dr\,d\varphi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r\cos^2\varphi+r\sin^2\varphi=r.$$

$$dx\,dy=r\,dr\,d\varphi$$

$$I=\iint\limits_{D}(x+y)\,dx\,dy=\iint\limits_{D'}\left(\frac{1}{2}+r\cos\varphi+\frac{1}{2}+r\sin\varphi\right)r\,dr\,d\varphi=\iint\limits_{D'}\left(r+r^2(\cos\varphi+\sin\varphi)\right)dr\,d\varphi$$

$$=\iint\limits_{D'}r\,dr\,d\varphi+\iint\limits_{D'}r^2(\cos\varphi+\sin\varphi)\,dr\,d\varphi,\qquad \iint\limits_{D'}r\,dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\frac{\sqrt{2}}{2}}r\,dr\right]d\varphi=2\pi\cdot\frac{1}{2}r^2\Big|_0^{\frac{\sqrt{2}}{2}}=\frac{\pi}{2}$$

$$\iint_{0} r^2(\cos\varphi+\sin\varphi)\,dr\,d\varphi=\int_0^{2/\sqrt{2}} r^2\left[\int_0^{2\pi}(\cos\varphi+\sin\varphi)\,d\varphi\right]dr=\frac{1}{3}r^3\Big|_0^{\frac{2}{\sqrt{2}}}\cdot\left(\sin\varphi\Big|_0^{2\pi}-\cos\varphi\Big|_0^{2\pi}\right)$$

$$=\frac{1}{3}\cdot\frac{8}{2\sqrt{2}}\left(0-(1-1)\right)=0$$

Prema tome:

$$\iint_0 (x+y)\,dx\,dy=\frac{\pi}{2}$$

Zadaci za vježbu

U zadacima 3525 — 3531 transformisati dvojni integral $\iint\limits_D f(x, y)\,dx\,dy$ na polarne koordinate ρ i φ ($x=\rho\cos\varphi$, $y=\rho\sin\varphi$), a zatim ga svesti na dvostvruki (sa određenim posebnim granicama integracije).

3525. D je krug: 1) $x^2+y^2<R^2$; 2) $x^2+y^2<ax$; 3) $x^2+y^2<by$.

3526. D je oblast ograničena kružnim linijama $x^2+y^2=4x$, $x^2+y^2=8x$ i pravama $y=x$ i $y=2x$.

3527. D je oblast koja predstavlja zajednički deo dva kruga $x^2+y^2<ax$ i $x^2+y^2<by$.

3528. D je oblast ograničena pravama $y=x$, $y=0$ i $x=1$.

3529. D je odsečak koji prava $x+y=2$ odseca od kruga $x^2+y^2=4$.

3530. D je oblast ograničena desnom petljom lemniskate $(x^2+y^2)^2=a^2(x^2-y^2)$.

3531. D je oblast određena nejednakostima $x>0$, $y\geqslant 0$, $(x^2+y^2)^3<4a^2x^2y^2$.

U zadacima 3532 — 3535 date dvostruke integrale transformisati na polarne koordinate.

3532. $\int_0^R dx\int_0^{\sqrt{R^2-x^2}} f(x, y)\,dy$. **3533.** $\int_{\frac{R}{2}}^{2R} dy\int_0^{\sqrt{2Ry-y^2}} f(x, y)\,dx$.

3534. $\int_0^R dx\int_0^{\sqrt{R^2-x^2}} f(x^2+y^2)\,dy$.

3535. $\int_0^{\frac{R}{\sqrt{1+R^2}}} dx\int_0^{Rx} f\left(\frac{y}{x}\right)dy+\int_{\frac{R}{\sqrt{1+R^2}}}^{R} dx\int_0^{\sqrt{R^2-x^2}} f\left(\frac{y}{x}\right)dy$.

U zadacima 3536 — 3540 izračunati date dvojne integrale prelazeći na polarne koordinate.

3536. $\iint\limits_D \ln(1+x^2+y^2)\,dx\,dy$, oblast D je četvrtina kruga $x^2+y^2<R^2$ koja leži u prvom kvadrantu.

3537. $\iint\limits_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$, oblast D je određena nejednakostima $x^2+y^2<1$, $x>0$, $y>0$.

3538. $\iint\limits_D (h-2x-3y)\,dx\,dy$, D je krug $x^2+y^2<R^2$.

3539. $\iint\limits_D \sqrt{R^2-x^2-y^2}\,dx\,dy$, D je krug $x^2+y^2<Rx$.

3540. $\iint\limits_D \operatorname{arctg}\frac{y}{x}\,dx\,dy$, D je deo prstena $x^2+y^2>1$, $x^2+y^2<9$, $y\geqslant\frac{x}{\sqrt{3}}$, $y<x\sqrt{3}$.

Rješenja

3525. 1) $\int_0^{2\pi} d\varphi\int_0^{R} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho$;

2) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\int_0^{a\cos\varphi} f(\rho\cos\varphi, \rho\sin\rho)\,\rho\,d\rho$;

3) $\int_0^{\pi} d\varphi\int_0^{b\sin\varphi} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho$.

3526. $\int_{\frac{\pi}{4}}^{\operatorname{arctg}2} d\varphi\int_{4\cos\varphi}^{8\cos\varphi} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho$.

3527. $\int_0^{\operatorname{arctg}\frac{a}{b}} d\varphi\int_0^{b\sin\varphi} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho+\int_{\operatorname{arctg}\frac{a}{b}}^{\frac{\pi}{2}} d\varphi\int_0^{a\cos\varphi} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho$.

3528. $\int_0^{\frac{\pi}{4}} d\varphi\int_0^{\sec\varphi} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho$.

3529. $\int_0^{\frac{\pi}{2}} d\varphi\int_{\sqrt{2}\sec\left(\varphi-\frac{\pi}{4}\right)}^{2} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho$.

3530. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi\int_0^{a\sqrt{\cos 2\varphi}} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho$.

3531. $\int_0^{\frac{\pi}{2}} d\varphi\int_0^{a\sin 2\varphi} f(\rho\cos\varphi), \rho\sin\varphi)\,\rho\,d\rho$.

3532. $\int_0^{\frac{\pi}{2}} d\varphi\int_0^{R} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho$.

3533. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\varphi\int_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho$.

3541. Na osnovu geometrijskih razmatranja pokazati da: ako se dekartove kordinate transformišu shodno obrascima $x=a\rho\cos\varphi$, $y=b\rho\sin\varphi$, u kojima su a i b konstante, onda će elemenat površine biti $d\sigma=ab\,\rho\,d\rho\,d\varphi$.

U zadacima 3542 — 3544 koristeći rezultat prethodnog zadatka i izabravši najpogodnije vrednosti za a i b, transformisati dvojne integrale.

3542. $\iint\limits_D f(x, y)\,dx\,dy$. D je oblast ograničena elipsom $\frac{x^2}{4}+\frac{y^2}{9}=1$.

3543. $\iint\limits_D f(x, y)\,dx\,dy$. D je oblast ograničena krivom $\left(x^2+\frac{y^2}{3}\right)^2=x^2 y$.

3544. $\iint\limits_D f\left(\sqrt{4-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)dx\,dy$, D je deo eliptičnog prstena ograničenog elipsama $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ i $\frac{x^2}{4a^2}+\frac{y^2}{4b^2}=1$, koji leži u prvom kvadrantu.

3545. Izračunati integral $\iint\limits_D xy\,dx\,dy$, u kojem je D oblast u prvom kvadrantu, ograničena elipsom $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

3546. Izračunati integral $\iint\limits_D \sqrt{xy}\,dx\,dy$, u kojem je D oblast u prvom kvadrantu, ograničena krivom $\left(\frac{x^2}{2}+\frac{y^2}{b}\right)^4=\frac{xy}{\sqrt{6}}$.

Rješenja

3534. $\frac{\pi}{2}\int\limits_0^R f(\rho^2)\,\rho\,\rho d.$ **3535.** $\frac{R^2}{2}\int\limits_0^{\operatorname{arctg} R} f(\operatorname{tg}\varphi)\,d\varphi.$

3536. $\frac{\pi}{4}[(1+R^2)\ln(1+R^2)-R^2]$. **3537.** $\frac{\pi(\pi-2)}{8}$. **3538.** $\pi R^2 h$.

3539. $\frac{R^3}{3}\left(\pi-\frac{4}{3}\right)$. **3540.** $\frac{\pi^2}{6}$.

3542. $x=2\rho\cos\varphi$, $y=3\rho\sin\varphi$; $I=6\int\limits_0^{2\pi} d\varphi\int\limits_0^1 f(2\rho\cos\varphi,\ 3\rho\sin\varphi)\,\rho\,d\rho.$

3543. $x=\rho\cos\varphi$, $y=\sqrt{3}\,\rho\sin\varphi$;

$$I=\sqrt{3}\int\limits_0^{\pi} d\varphi \int\limits_0^{\sqrt{3}\cos^2\varphi\sin\varphi} f(\rho\cos\varphi,\ \sqrt{3}\,\rho\sin\varphi)\,\rho\,d\rho.$$

3544. $x=a\rho\cos\varphi$, $y=b\rho\sin\varphi$; $I=ab\int\limits_0^{\frac{\pi}{2}} d\varphi\int\limits_1^2 f(\sqrt{4-\rho^2})\,\rho\,d\rho.$

3545. $\frac{a^2b^2}{8}$. **3546.** $\frac{1}{\frac{4}{\sqrt{6}}}$.